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Maxwell's Equations**

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Abstract

By means of the limit and jump relations of classical potential theory with respect to the vectorial Helmholtz equation a wavelet approach is established on a regular surface. The multiscale procedure is constructed in such a way that the emerging scalar, vectorial and tensorial potential kernels act as scaling functions. Corresponding wavelets are defined via a canonical refinement equation. A tree algorithm for fast decomposition of a complex-valued vector field given on a regular surface is developed based on numerical integration rules. By virtue of this tree algorithm, an efficient numerical method for the solution of vectorial Fredholm integral equations on regular surfaces is discussed in more detail. The resulting multiscale formulation is used to solve boundary-value problems for the time harmonic Maxwell's equations corresponding to regular surfaces.

Key words. regular surface, potential operators, limit and jump relations, scaling functions and wavelets, pyramid scheme, time-harmonic Maxwell's equations

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1 Introduction

During the last two decades, wavelets have become the mathematical tool of choice for breaking up a complicated function into many simple pieces. The importance of wavelets is based on the fact that they are 'building blocks' that enable fast decorrelation of huge data sets. In consequence, three features are incorporated in this way of thinking of wavelets, namely the basis property, decorrelation and fast computation. Basically, wavelet analysis works by convolving the function under consideration against 'dilated' and 'shifted' versions of one fixed function, viz., the 'mother wavelet'. Traditionally, applications of wavelets have been signal analysis, image processing, denoising, etc. But there is also a growing interest in the numerical treatment of partial differential equations. However, strategies of extending the applicability of wavelet techniques to boundary-value problems corresponding to regular surfaces have only rarely attempted. One promising approach is to construct wavelets from the limit and jump relations of the layer potentials used as ansatz for transforming the differential equation under consideration into an integral equation. For the Laplace equation in \mathbb{R}^3 this approach has been developed in [12] and for the scalar Helmholtz equation in [13]. As a matter of fact, this concept offers multilevel approximation of functions on regular surfaces by interpreting the emerging kernel functions as scaling functions within a scale continuous wavelet framework. In doing so, the scale discretized approach follows by a canonical discretization procedure on the regular surface by means of numerical integration techniques.

In this paper, we propose analogous techniques based on layer potential operators associated to the vectorial Helmholtz equation (i.e., time-harmonic Maxwell's equations) for both the generation of complex-valued scalar, vectorial and tensorial wavelets on regular surfaces and a multiscale approach for the solution of vectorial integral equations corresponding to the vectorial Helmholtz equation. Instead of applying a conventional wavelet construction oriented on Euclidean theory for discretizing the integral equations in accordance with a collocational, Galerkin or least squares procedure we use the occurring kernels of the layer potentials themselves to establish a new class of vectorial wavelets on regular surfaces. In consequence, a new wavelet approach for complex-valued vector fields naturally grows out of the layer potentials of the vectorial Helmholtz equation.

For the purpose of fast computation, our paper shows how a pyramid scheme can be developed adequately within the setup of multiresolution analysis. In similar way, a new method for the solution of the integral equations involving the time-harmonic Maxwell's equations can be formulated and implemented. Our approach is written down in such a way that not only complex vector fields on regular surfaces can be approximated efficiently using vectorial wavelet transforms generated by Helmholtz layer potentials, but also both single and double layer potentials themselves (and their combinations occurring in integral equations corresponding to vectorial boundary-value problems) allow a canonical multiscale reconstruction and decomposition within a pyramid scheme. Boundary integral equations of singular type are replaced approximately by a resulting boundary integral equation of regular type. In doing so the scale parameter acts as an indicator for regularization.

The outline of this paper is as follows: First we introduce the notation that is needed for our wavelet approach. We define regular surfaces in three-dimensional Euclidean space \mathbb{R}^3 on which our whole theory is established. Next, we introduce the classical potential operators with respect to the vectorial Helmholtz differential equation. Their kernels are the constituting ingredients of this work. We list the limit and jump relations of the potential operators and their dual counterparts formulated in the framework of both the Banach space of continuous functions as well as the Hilbert space of square-integrable functions on regular surfaces. The setup of a vectorial multiresolution analysis (i.e., scaling functions, scale spaces, wavelets, detail spaces) is defined by interpreting the kernel functions of the limit and jump integral operators as scaling functions on regular surfaces. The distance to the parallel surfaces of the regular surface under consideration defines the scale level of low-pass filtering by the scaling function. As a particularly important example, we derive the spectral representation of the vectorial scaling function when restricted to the sphere. After that we come to a scale discretization of our approach. As a significant aspect of scientific computing we develop a pyramid scheme providing a fast wavelet transform (FWT). Finally, we adapt a modification of this scheme to the vectorial boundary integral equations occurring in the solution theory of the time-harmonic Maxwell's equations.

2 Preliminaries

At first we want to introduce some basic notations which we need for our wavelet approach. As usual, \mathbb{R}^3 denotes three-dimensional Euclidian space. For $x, y \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, $y = (y_1, y_2, y_3)^T$ the inner product $x \cdot y$ and the tensor product $x \otimes y$ are defined by

$$x \cdot y = x^T y = \sum_{i=1}^3 x_i y_i, \quad x \otimes y = xy^T = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}.$$

For elements $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, different from the origin, we have

$$x = r\xi, \quad r = |x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of x . The unit sphere in \mathbb{R}^3 is denoted by Ω . If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in \mathbb{R}^3 , the points $\xi \in \Omega$ may be represented in polar coordinates by

$$\begin{aligned} \xi &= t\varepsilon^3 + \sqrt{1-t^2} (\cos \varphi \varepsilon^1 + \sin \varphi \varepsilon^2) \\ t &= \cos \vartheta, \vartheta \in [0, \pi], \varphi \in [0, 2\pi] \end{aligned}$$

Next we introduce some settings which are standard in potential theory (see, for example, [17], [21], [25], [26]). We begin our considerations by introducing the notation of a regular surface:

DEFINITION 2.1 *A surface $\Sigma \subset \mathbb{R}^3$ is called regular, if it satisfies the following properties:*

- (i) Σ divides the three-dimensional Euclidean space \mathbb{R}^3 into the bounded region Σ_{int} (inner space) and the unbounded region Σ_{ext} (outer space) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$,
- (ii) Σ_{int} contains the origin,
- (iii) Σ is a closed and compact surface free of double points,
- (iv) Σ has a continuously differentiable unit normal field ν pointing into the outer space Σ_{ext} .

Geoscientifically regular surfaces Σ are, for example, sphere, ellipsoid, spheroid, geoid, (regular) Earth's surface.

Given a regular surface, then there exist positive constants α, β such that

$$\alpha < \sigma^{\text{inf}} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{\text{sup}} < \beta.$$

As usual, $A_{\text{int}}, B_{\text{int}}$ (resp. $A_{\text{ext}}, B_{\text{ext}}$) denote the inner (resp. outer) space of the sphere A (resp. B) around the origin with radius α (resp. β). $\Sigma_{\text{int}}^{\text{inf}}, \Sigma_{\text{int}}^{\text{sup}}$ (resp. $\Sigma_{\text{ext}}^{\text{inf}}, \Sigma_{\text{ext}}^{\text{sup}}$) denote the inner (resp. outer) space of the sphere Σ^{inf} (resp. Σ^{sup}) around the origin with radius σ^{inf} (resp. σ^{sup}).

The set

$$\Sigma(\tau) = \{x \in \mathbb{R}^3 | x = y + \tau\nu(y), y \in \Sigma\}$$

generates a parallel surface which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. It is well known from differential geometry (see, e.g., [24]) that if $|\tau|$ is sufficiently small, then the surface $\Sigma(\tau)$ is regular, and the normal to one parallel surface is a normal to the other. According to our regularity assumptions imposed on Σ the functions

$$\begin{aligned} (x, y) &\mapsto \frac{|\nu(x) - \nu(y)|}{|x - y|}, \quad (x, y) \in \Sigma \times \Sigma, x \neq y \\ (x, y) &\mapsto \frac{|\nu(x) \cdot (x - y)|}{|x - y|^2}, \quad (x, y) \in \Sigma \times \Sigma, x \neq y \end{aligned}$$

are bounded. Hence, there exists a constant $M > 0$ such that

$$\begin{aligned} |\nu(x) - \nu(y)| &\leq M|x - y|, \\ |\nu(x) \cdot (x - y)| &\leq M|x - y|^2, \end{aligned}$$

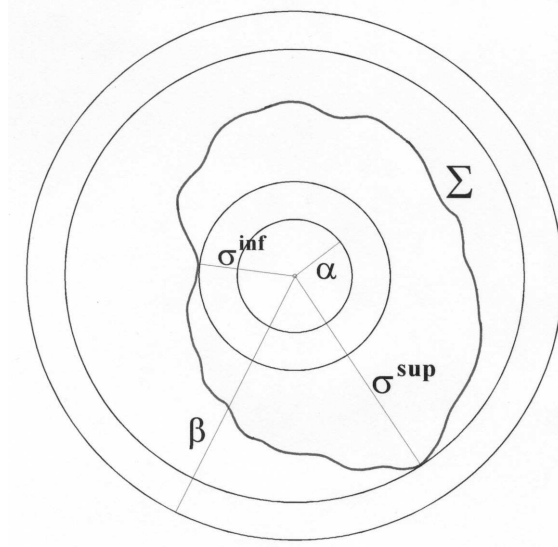


Figure 1: Regular surface (geometrical illustration)

for all $(x, y) \in \Sigma \times \Sigma$. Moreover, it is easy to see that

$$\inf_{x, y \in \Sigma} |x + \tau \nu(x) - (y + \sigma \nu(y))| = |\tau - \sigma|$$

provided that $|\tau|, |\sigma|$ are sufficiently small.

A variety of function spaces will be needed in this article. Let $C(U)$ be the set of all continuous, complex-valued functions defined on a set $U \subset \mathbb{R}^3$, equipped with the standard supremum norm. A function is said to be of class $C^{(k)}(U)$, $0 \leq k \leq \infty$, if it is k -times continuously differentiable on U . If $U \subset \mathbb{R}^3$ is a measurable subset of \mathbb{R}^3 , the set of scalar functions $F : U \rightarrow \mathbb{R}$ which are measurable and for which

$$\|F\|_{L^p(U)} = \left(\int_U |F(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

is denoted by $L^p(U)$, where dx denotes the volume element in U . Note that in the case of $U = \Sigma$ where Σ is a regular surface we write $d\omega_\Sigma(x)$ (or simply $d\omega(x)$) instead of dx and $d\omega_R(x)$ instead of $d\omega_{\Omega_R}(x)$ if Σ is a sphere with radius $R > 0$.

In analogy to the scalar case we define function spaces of vector valued functions. These spaces will normally be symbolized by lower-case letters. Let $c(U)$ be the set of all vector valued, continuous functions $f : U \rightarrow \mathbb{R}^3$ defined on the set $U \subset \mathbb{R}^3$, equipped with the norm $\|f\|_{c(U)} = \sup_{x \in U} |f(x)|$. A vector field f is said to be of class $c^{(k)}(U)$, $0 \leq k \leq \infty$, if every component function $f \cdot \varepsilon^i$, $i = 1, \dots, n$, of f is k -times continuously differentiable on U . The set of vector fields $f : U \rightarrow \mathbb{R}$ which are measurable and for which

$$\|f\|_{l^p(U)} = \left(\int_U |f(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

is denoted by $l^p(U)$. If $U = \Sigma$, the space $c^{(k)}(\Sigma)$ can be split naturally into a subspace $c_{tan}^{(k)}(\Sigma)$ containing vector fields which are purely tangential to the regular surface Σ and a subspace $c_{nor}^{(k)}(\Sigma)$ which contains purely normal fields, i.e.,

$$c^{(k)}(\Sigma) = c_{tan}^{(k)}(\Sigma) \oplus c_{nor}^{(k)}(\Sigma).$$

The spaces $l_{tan}^2(\Sigma)$ and $l_{nor}^2(\Sigma)$ are the completion of $c_{tan}^{(0)}(\Sigma)$ resp. $c_{nor}^{(0)}(\Sigma)$ with respect to the $\|\cdot\|_{l^2(\Sigma)}$ -norm, i.e.,

$$\overline{c_{tan}^{(0)}(\Sigma)}^{\|\cdot\|_{l^2(\Sigma)}} = l_{tan}^2(\Sigma), \quad \overline{c_{nor}^{(0)}(\Sigma)}^{\|\cdot\|_{l^2(\Sigma)}} = l_{nor}^2(\Sigma).$$

3 Limit Formulae and Jump Relations

DEFINITION 3.1 *Let $k \in \mathbb{R} \setminus \{0\}$ and $G_k : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be the fundamental solution of the scalar Helmholtz equation given by*

$$G_k(x, y) = \frac{\exp(ik|x-y|)}{|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y.$$

For $k \in \mathbb{R} \setminus 0$, $\tau \neq \sigma$ with $|\tau|$ and $|\sigma|$ sufficiently small we define the operators $P(\tau, \sigma; k) : l^2_{tan}(\Sigma) \rightarrow c^\infty_{tan}(\Sigma)$, $P_{curl}(\tau, \sigma; k) : l^2_{tan}(\Sigma) \rightarrow c^\infty_{tan}(\Sigma)$ and $Q(\tau, \sigma; k) : l^2_{tan}(\Sigma) \rightarrow c^\infty_{tan}(\Sigma)$ by

$$\begin{aligned} P(\tau, \sigma; k)f(x) &= \int_{\Sigma} f(y)G_k(x + \tau\nu(x), y + \sigma\nu(y))d\omega(y), \\ P_{curl}(\tau, \sigma; k)f(x) &= \nu(x) \wedge \int_{\Sigma} f(y) \wedge (\nabla_y G_k)(x + \tau\nu(x), y + \sigma\nu(y))d\omega(y), \\ Q(\tau, \sigma; k)f(x) &= \nu(x) \wedge \int_{\Sigma} \left(f(y)G_k + ((f(y)\nabla_y)\nabla_y G_k) \right)(x + \tau\nu(x), y + \sigma\nu(y))d\omega(y) \\ &= \nu(x) \wedge \int_{\Sigma} \left(\nabla_y \otimes \nabla_y + I \right) G_k(x + \tau\nu(x), y + \sigma\nu(y))f(y)d\omega(y). \end{aligned}$$

3.1 Formulation in $(c(\Sigma), \|\cdot\|_{c(\Sigma)})$

For $f \in c_{tan}(\Sigma)$, the functions $P(\tau, \sigma; k)f$, $P_{curl}(\tau, \sigma; k)f$ and $Q(\tau, \sigma; k)f$ can be continued continuously to the surface Σ , but the limits depend from which parallel surface (inner or outer) the points x tend to Σ . On the other hand, the functions $P(\tau, \sigma; k)f$, $P_{curl}(\tau, \sigma; k)$ and $Q(\tau, \sigma; k)f$, also are defined on the surface Σ , i.e., the integrals exist and are continuous for $\tau = \sigma = 0$. However, the integrals do not coincide, in general, with the inner or outer limits of the potentials (cf. [1]).

If $\tau = \sigma = 0$, the kernels of the potentials have weak singularities. The integrals formally defined by

$$\begin{aligned} P(0, 0; k)f(x) &= \int_{\Sigma} f(y) \frac{\exp(ik|x-y|)}{|x-y|} d\omega(y), \\ P_{curl}(0, 0; k)f(x) &= \nu(x) \wedge \int_{\Sigma} f(y) \wedge (x-y) \frac{\exp(ik|x-y|)}{|x-y|} \left(ik - \frac{1}{|x-y|} \right) d\omega(y), \end{aligned}$$

and

$$Q(0, 0; k)f(x) = \nu(x) \wedge \int_{\Sigma} f(y)G_k(x, y) + ((f(y)\nabla_y)\nabla_y G_k)(x, y)d\omega(y)$$

however, exist and define linear bounded operators in $l^2(\Sigma)$. $P(0, 0; k)$, $P_{curl}(0, 0; k)$ and $Q(0, 0; k)$ map $c^{(0)}_{tan}(\Sigma)$ into itself (see [23, 25]). Furthermore, the operators are continuous (even compact) with respect to $\|\cdot\|_{c^{(0)}(\Sigma)}$.

The operators $P(\tau, \sigma; k)^*$, $P_{curl}(\tau, \sigma; k)^*$ and $Q(\tau, \sigma; k)^*$ satisfying

$$\begin{aligned} (f, P(\tau, \sigma; k)g)_{l^2(\Sigma)} &= (P(\tau, \sigma; k)^*f, g)_{l^2(\Sigma)} \\ (f, P_{curl}(\tau, \sigma; k)g)_{l^2(\Sigma)} &= (P_{curl}(\tau, \sigma; k)^*f, g)_{l^2(\Sigma)} \\ (f, Q(\tau, \sigma; k)g)_{l^2(\Sigma)} &= (Q(\tau, \sigma; k)^*f, g)_{l^2(\Sigma)} \end{aligned}$$

for all $f, g \in l^2(\Sigma)$ are called the *adjoint operator* of $P(\tau, \sigma; k)$, $P_{curl}(\tau, \sigma; k)$ and $Q(\tau, \sigma; k)$ with respect to $l^2(\Sigma)$ -inner-product.

A relation between the operators $P(\tau, \sigma; k)$, $P_{curl}(\tau, \sigma; k)$ and $Q(\tau, \sigma; k)$ and their adjoint operators is given by the following result.

LEMMA 3.2 *For $\tau \neq \sigma$ and $k \in \mathbb{R} \setminus 0$, let $P^*(\tau, \sigma; k)$ and $Q^*(\tau, \sigma; k)$ be the adjoint operators of $P(\tau, \sigma; k)$, resp. $Q(\tau, \sigma; k)$ with respect to the $l^2(\Sigma)$ -inner-product. Then we have*

$$\begin{aligned} P^*(\tau, \sigma; k) &= TP(\sigma, \tau; -k)T \\ Q^*(\tau, \sigma; k) &= TQ(\sigma, \tau; -k)T \end{aligned}$$

where the tangential projection operator T is the canonical extension of the operator $T : c(\Sigma) \rightarrow c_{tan}(\Sigma)$ given by $Tf(x) = \nu(x) \wedge (f(x) \wedge \nu(x))$, $x \in \Sigma$, to $l^2(\Sigma)$.

The lemma can be proved by a straightforward calculation.

The potential operators now enable us to give concise formulations of the classical limit formulae and jump relations in potential theory. Let I be the identity operator in $l^2(\Sigma)$. Suppose that, for all sufficiently small values $\tau > 0$, $L_i^\pm(\tau)$, $i = 1, 2, 3$ and $J_i(\tau)$, $i = 1, \dots, 5$, respectively, define the following operators:

$$\begin{aligned} L_1^\pm(\tau; k) &= P(\pm\tau, 0; k) - P(0, 0; k), \\ L_2^\pm(\tau; k) &= P_{curl}(\pm\tau, 0; k) - P_{curl}(0, 0; k) \pm 2\pi I \\ L_3^\pm(\tau; k) &= Q(\pm\tau, 0; k) - Q(0, 0), \end{aligned}$$

$$\begin{aligned} J_1(\tau; k) &= P(\tau, 0; k) - P(-\tau, 0; k) \\ J_2(\tau; k) &= P_{curl}(\tau, 0; k) - P_{curl}(-\tau, 0; k) + 4\pi I, \\ J_3(\tau; k) &= Q(\tau, 0; k) - Q(-\tau, 0; k), \\ J_4(\tau; k) &= P(\tau, 0; k) + P(-\tau, 0; k) - 2P(0, 0; k), \\ J_5(\tau; k) &= P_{curl}(\tau, 0; k) + P_{curl}(-\tau, 0; k) - 2P_{curl}(0, 0; k). \end{aligned}$$

Then, for $f \in c_{tan}(\Sigma)$, the limit and jump relations of classical potential theory may be formulated by

$$\begin{aligned} \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|L_i^\pm(\tau)f\|_{c^{(0)}(\Sigma)} &= 0, & \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|J_j(\tau)f\|_{c^{(0)}(\Sigma)} &= 0, \\ \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|L_i^\pm(\tau)^*f\|_{c^{(0)}(\Sigma)} &= 0, & \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|J_j(\tau)^*f\|_{c^{(0)}(\Sigma)} &= 0; \end{aligned} \tag{1}$$

for $i = 1, 2, 3$, and $j = 1, \dots, 5$.

3.2 Formulation in $(l^2(\Sigma), \|\cdot\|_{l^2(\Sigma)})$

The limit and jump relations (1) can be generalized to the Hilbert space $l^2(\Sigma)$ as follows:

THEOREM 3.3 For all $f \in l_{tan}^2(\Sigma)$, $i = 1, 2, 3$, and $j = 1, \dots, 5$

$$\begin{aligned} \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|L_i^\pm(\tau)f\|_{l^2(\Sigma)} &= 0, & \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|J_j(\tau)f\|_{l^2(\Sigma)} &= 0, \\ \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|L_i^\pm(\tau)^*f\|_{l^2(\Sigma)} &= 0, & \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|J_j(\tau)^*f\|_{l^2(\Sigma)} &= 0. \end{aligned}$$

Proof. Denote by $T(\tau)$ one of the operators $L_i^\pm(\tau)$, $i = 1, 2, 3$, $J_j(\tau)$, $j = 1, \dots, 5$. Then, by virtue of the norm estimate,

$$\|f\|_{l^2(\Sigma)} < \sqrt{\|\Sigma\|} \|f\|_{c^{(0)}(\Sigma)}, \quad \|\Sigma\| = \int_{\Sigma} d\omega, \quad f \in c^{(0)}(\Sigma), \tag{2}$$

we obtain

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|T(\tau)f\|_{l^2(\Sigma)} = 0, \quad \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|T(\tau)^*f\|_{l^2(\Sigma)} = 0$$

for all $f \in c_{tan}(\Sigma)$. Therefore there exists a constant $D(f) > 0$ such that

$$\|T(\tau)f\|_{L^2(\Sigma)} < D(f), \quad \|T(\tau)^*f\|_{L^2(\Sigma)} < D(f)$$

for all $\tau \leq \tau_0$ (τ_0 sufficiently small). The uniform boundedness principle of functional analysis (see, e.g., [16]) then shows us that there exists a constant $M > 0$ such that

$$\left\| T(\tau)|_{c^{(0)}(\Sigma)} \right\|_{l^2(\Sigma)} \leq M, \quad \left\| T(\tau)^*|_{c^{(0)}(\Sigma)} \right\|_{l^2(\Sigma)} \leq M$$

for all $\tau \leq \tau_0$. The operators $(T(\tau)^*T(\tau))$ are self-adjoint, and their restrictions to the Banach space $c^{(0)}(\Sigma)$ are continuous. We now modify a technique due to [20]. According to the Cauchy-Schwarz inequality we get for $f \in c_{tan}^{(0)}(\Sigma)$

$$\begin{aligned} \left(\|T(\tau)f\|_{l^2(\Sigma)} \right)^2 &= (T(\tau)f, T(\tau)f)_{l^2(\Sigma)} \\ &= (f, (T(\tau)^*T(\tau))f)_{l^2(\Sigma)} \\ &\leq \|f\|_{l^2(\Sigma)} \|(T(\tau)^*T(\tau))f\|_{l^2(\Sigma)}. \end{aligned}$$

Consequently it follows that

$$\begin{aligned} \left(\|T(\tau)f\|_{l^2(\Sigma)} \right)^{2^2} &\leq \left(\|f\|_{l^2(\Sigma)} \right)^2 \left(\|T(\tau)^*T(\tau)f\|_{l^2(\Sigma)} \right)^2 \\ &\leq \left(\|f\|_{l^2(\Sigma)} \right)^2 \|f\|_{l^2(\Sigma)} \|(T(\tau)^*T(\tau))^2 f\|_{l^2(\Sigma)}. \end{aligned}$$

Induction yields

$$\left(\|T(\tau)f\|_{l^2(\Sigma)} \right)^{2^n} \leq \left(\|f\|_{l^2(\Sigma)} \right)^{2^{n-1}} \left\| (T(\tau)^*T(\tau))^{2^{n-1}} f \right\|_{l^2(\Sigma)}$$

for all positive integers n . According to the norm estimate (2) and the boundedness of the operators $T(\tau), T(\tau)^*$ for all $\tau \leq \tau_0$ there exists a positive constant K such that

$$\left(\|T(\tau)f\|_{l^2(\Sigma)} \right)^{2^n} \leq \sqrt{\|\Sigma\|} K^{2^n} \left(\|f\|_{l^2(\Sigma)} \right)^{2^{n-1}} \|f\|_{c^{(0)}(\Sigma)}.$$

Therefore, for positive integers n and all $f \in c_{tan}(\Sigma)$ with $f \neq 0$, we find

$$\frac{\|T(\tau)f\|_{l^2(\Sigma)}}{\|f\|_{l^2(\Sigma)}} \leq K \left(\frac{\sqrt{\|\Sigma\|} \|f\|_{c(\Sigma)}}{\|f\|_{l^2(\Sigma)}} \right)^{2^{-n}}.$$

Letting n tend to infinity we obtain for all $f \neq 0$

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt{\|\Sigma\|} \|f\|_{c^{(0)}(\Sigma)}}{\|f\|_{l^2(\Sigma)}} \right)^{2^{-n}} = 1.$$

This shows us that the norm $\|T(\tau)\|_{l^2(\Sigma)}$ of the operator $T(\tau), \tau \leq \tau_0$, can be estimated by K , i.e.,

$$\|T(\tau)f\|_{l^2(\Sigma)} \leq K \|f\|_{l^2(\Sigma)} \quad (3)$$

for all $f \in c_{tan}^{(0)}(\Sigma)$ and all $\tau \leq \tau_0$. The same argument holds true for the adjoint operators, i.e.,

$$\|T(\tau)^*f\|_{l^2(\Sigma)} \leq K \|f\|_{l^2(\Sigma)} \quad (4)$$

for all $f \in c_{tan}^{(0)}(\Sigma)$ and all $\tau \leq \tau_0$. The space $c_{tan}^{(0)}(\Sigma)$ is as a linear subspace dense in $l_{tan}^2(\Sigma)$. Thus, by the Hahn-Banach theorem (cf. [16]), we can extend the operators $T(\tau)$ and $T(\tau)^*$ from $c_{tan}^{(0)}(\Sigma)$ to $l_{tan}^2(\Sigma)$ without enlarging their norms.

Now let $f \in l_{tan}^2(\Sigma)$ be given, then there exists a sequences of functions $\{f_n\}_{n \in \mathbb{N}} \subset c_{tan}^{(0)}(\Sigma)$, such that for every $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ with

$$\|f - f_n\|_{l^2(\Sigma)} < \epsilon,$$

for all $n \geq N(\epsilon)$. Then we have using Eq. (3) and Eq. (4)

$$\begin{aligned} 0 \leq \|T(\tau)f\|_{l^2(\Sigma)} &\leq \|T(\tau)f_n\|_{l^2(\Sigma)} + \|T(\tau)(f - f_n)\|_{l^2(\Sigma)} \\ &\leq \|T(\tau)f_n\|_{l^2(\Sigma)} + K \|f - f_n\|_{l^2(\Sigma)} \end{aligned}$$

and, thus, by using $f \in c_{tan}^{(0)}(\Sigma)$

$$0 < \lim_{\tau \rightarrow 0} \|T(\tau)f\|_{l^2(\Sigma)} \leq D \|f - f_n\|_{l^2(\Sigma)}$$

for all $n \in \mathbb{N}$. The limit $n \rightarrow \infty$ now yields the desired result. \square

4 Multiscale Modelling in $(l^2(\Sigma), \|\cdot\|_{l^2(\Sigma)})$

Writing out the limit and jump relations (Theorem 3.3) explicitly we obtain the following corollary.

COROLLARY 4.1 For $f \in l^2_{tan}(\Sigma)$

$$\begin{aligned} \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Sigma} \Phi_{\tau}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} P(0, 0; k) f & , \quad i = 1, 2 \\ 0 & , \quad i = 3, \end{cases} \\ \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \nu(\cdot) \wedge \int_{\Sigma} \varphi_{\tau}^i(\cdot, y) \wedge f(y) d\omega(y) &= \begin{cases} f & , \quad i = 4, 5 \\ P_{curl}(0, 0; k) f & , \quad i = 6, \end{cases} \\ \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \nu(\cdot) \wedge \int_{\Sigma} \Phi_{\tau}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} Q(0, 0; k) f & , \quad i = 7, 8, \\ 0 & , \quad i = 9 \end{cases} \end{aligned}$$

holds in the sense of the $\|\cdot\|_{l^2(\Sigma)}$ -norm, where the scalar kernel functions $\Phi_{\tau}^i(x, y)$ with $x, y \in \Sigma$ and $i = 1, 2, 3$ are given by

$$\begin{aligned} \Phi_{\pm\tau}^1(x, y) &= \frac{\exp(ik|x \pm \tau\nu(x) - y|)}{|x \pm \tau\nu(x) - y|}, \\ \Phi_{\tau}^2(x, y) &= \frac{1}{2} \left(\frac{\exp(ik|x + \tau\nu(x) - y|)}{|x + \tau\nu(x) - y|} + \frac{\exp(ik|x - \tau\nu(x) - y|)}{|x - \tau\nu(x) - y|} \right), \\ \Phi_{\tau}^3(x, y) &= \left(\frac{\exp(ik|x + \tau\nu(x) - y|)}{|x + \tau\nu(x) - y|} - \frac{\exp(ik|x - \tau\nu(x) - y|)}{|x - \tau\nu(x) - y|} \right). \end{aligned}$$

The vectorial kernel functions $\varphi_{\tau}^i(x, y)$ with $x, y \in \Sigma$ and $i = 4, 5, 6$ are given by

$$\begin{aligned} \varphi_{\pm\tau}^4(x, y) &= \frac{1}{2\pi} \left((x + \tau\nu(x) - y) \frac{\exp(ik|x + \tau\nu(x) - y|)}{|x + \tau\nu(x) - y|^2} \left(ik - \frac{1}{|x + \tau\nu(x) - y|} \right) \right. \\ &\quad \left. - (x - y) \frac{\exp(ik|x - y|)}{|x - y|^2} \left(ik - \frac{1}{|x - y|} \right) \right), \\ \varphi_{\tau}^5(x, y) &= \frac{1}{4\pi} \left((x + \tau\nu(x) - y) \frac{\exp(ik|x + \tau\nu(x) - y|)}{|x + \tau\nu(x) - y|^2} \left(ik - \frac{1}{|x + \tau\nu(x) - y|} \right) \right. \\ &\quad \left. - (x - \tau\nu(x) - y) \frac{\exp(ik|x - \tau\nu(x) - y|)}{|x - \tau\nu(x) - y|^2} \left(ik - \frac{1}{|x - \tau\nu(x) - y|} \right) \right), \\ \varphi_{\tau}^6(x, y) &= \frac{1}{2} \left((x + \tau\nu(x) - y) \frac{\exp(ik|x + \tau\nu(x) - y|)}{|x + \tau\nu(x) - y|^2} \left(ik - \frac{1}{|x + \tau\nu(x) - y|} \right) \right. \\ &\quad \left. + (x - \tau\nu(x) - y) \frac{\exp(ik|x - \tau\nu(x) - y|)}{|x - \tau\nu(x) - y|^2} \left(ik - \frac{1}{|x - \tau\nu(x) - y|} \right) \right). \end{aligned}$$

The tensorial kernel functions $\Phi_{\tau}^i(x, y)$ with $x, y \in \Sigma$ and $i = 7, 8, 9$ are given by

$$\begin{aligned} \Phi_{\pm\tau}^7(x, y) &= (\nabla_y \otimes \nabla_y + I) G_k(x \pm \tau\nu(x), y), \\ &= \frac{\exp(ik|x \pm \tau\nu(x) - y|)}{|x \pm \tau\nu(x) - y|} \left[\left(\frac{3}{|x \pm \tau\nu(x) - y|^3} - \frac{ik}{|x \pm \tau\nu(x) - y|^2} - \frac{k^2}{|x \pm \tau\nu(x) - y|} \right) (y \otimes y) \right. \\ &\quad \left. + \left(\frac{1}{|x \pm \tau\nu(x) - y|} - ik + 1 \right) I \right], \\ \Phi_{\tau}^8(x, y) &= \frac{1}{2} ((\nabla_y \otimes \nabla_y + I) G_k(x + \tau\nu(x), y) + (\nabla_y \otimes \nabla_y + I) G_k(x - \tau\nu(x), y)), \\ \Phi_{\tau}^9(x, y) &= (\nabla_y \otimes \nabla_y + I) G_k(x + \tau\nu(x), y) - (\nabla_y \otimes \nabla_y + I) G_k(x - \tau\nu(x), y). \end{aligned}$$

Here, I stands for the identity operator as well as for the identity matrix.

It should be noted, that explicit representation of the tensorial kernel functions $\Phi_\tau^i(x, y)$, $i = 8, 9$, can be calculated as well. They are not given here for simplicity.

4.1 Scaling and Wavelet Functions

DEFINITION 4.2 For $\tau > 0$ and $i \in \{1, 2, 3\}$, the family $\{\Phi_\tau^i\}_{\tau>0}$ of kernels $\Phi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is called a scalar Σ -scaling function of type i . Moreover, $\Phi_1^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ (i.e., $\tau = 1$) is called the scalar mother kernel of the scalar Σ -scaling function of type i .

For $\tau > 0$ and $i \in \{4, 5, 6\}$, the family $\{\varphi_\tau^i\}_{\tau>0}$ of kernels $\varphi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is called a vectorial Σ -scaling function of type i . Moreover, $\varphi_1^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ (i.e., $\tau = 1$) is called the vectorial mother kernel of the vectorial Σ -scaling function of type i .

For $\tau > 0$ and $i \in \{7, 8, 9\}$, the family $\{\Phi_\tau^i\}_{\tau>0}$ of kernels $\Phi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is called a tensorial Σ -scaling function of type i . Moreover, $\Phi_1^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ (i.e., $\tau = 1$) is called the tensorial mother kernel of the tensorial Σ -scaling function of type i .

Correspondingly, for $\tau > 0$, families of scalar, vectorial and tensorial wavelets are defined as follows.

DEFINITION 4.3 For $\tau > 0$ and $i \in \{1, 2, 3\}$, the family $\{\Psi_\tau^i\}_{\tau>0}$ of scalar kernels $\Psi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ given by

$$\Psi_\tau^i(x, y) = -\alpha(\tau)^{-1} \frac{d}{d\tau} \Phi_\tau^i(x, y), \quad x, y \in \Sigma, \quad (5)$$

is called a scalar Σ -wavelet function of type i .

For $\tau > 0$ and $i \in \{4, 5, 6\}$, the family $\{\psi_\tau^i\}_{\tau>0}$ of vectorial kernels $\psi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ given by

$$\psi_\tau^i(x, y) = -\alpha(\tau)^{-1} \frac{d}{d\tau} \varphi_\tau^i(x, y), \quad x, y \in \Sigma, \quad (6)$$

is called a vectorial Σ -wavelet function of type i .

For $\tau > 0$ and $i \in \{7, 8, 9\}$, the family $\{\Psi_\tau^i\}_{\tau>0}$ of tensorial kernels $\Psi_\tau^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ given by

$$\Psi_\tau^i(x, y) = -\alpha(\tau)^{-1} \frac{d}{d\tau} \Phi_\tau^i(x, y), \quad x, y \in \Sigma, \quad (7)$$

is called a tensorial Σ -wavelet function of type i .

In the remainder of this paper we particularly choose $\alpha(\tau) = \tau^{-1}$ (of course, other weight functions than $\alpha(\tau) = \tau^{-1}$ can be chosen in (5), (6) and (7). The differential equations (5), (6) and (7) are called the (scale continuous) Σ -scaling equation of type i .

DEFINITION 4.4 Let $\{\Phi_\tau^i\}_{\tau>0}$, $\{\varphi_\tau^i\}_{\tau>0}$ and $\{\Phi_\tau^i\}_{\tau>0}$ be a scalar, vectorial resp. tensorial Σ -scaling function of type i . Then the associated scalar, vectorial and tensorial Σ -wavelet transforms of type i are defined by

$$(WT)^{(i)} : l_{tan}^2(\Sigma) \rightarrow l^2((0, \infty) \times \Sigma)$$

$$(WT)^{(i)}(f)(\tau, x) = \begin{cases} \int_{\Sigma} \Psi_\tau^i(x, y) f(y) d\omega(y) & , \quad i = 1, 2, 3 \\ \nu(x) \wedge \int_{\Sigma} \psi_\tau^i(x, y) \wedge f(y) d\omega(y) & , \quad i = 4, 5, 6 \\ \nu(x) \wedge \int_{\Sigma} \Psi_\tau^i(x, y) f(y) d\omega(y) & , \quad i = 7, 8, 9. \end{cases}$$

For simplicity we omit at this point the explicit representation of the Σ -wavelet functions but the reader should note that they can be calculated.

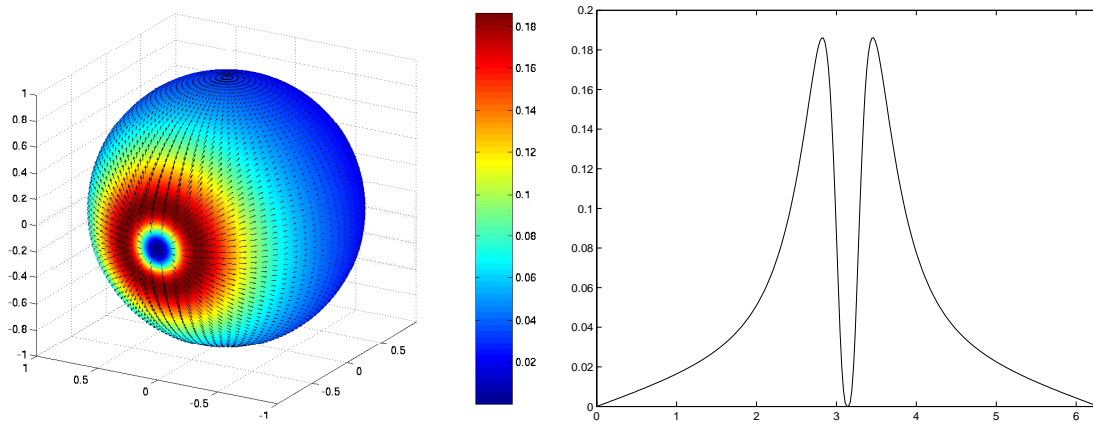


Figure 2: Real part of the vectorial Σ -scaling-function φ_τ^5 for $\tau = 0.25$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

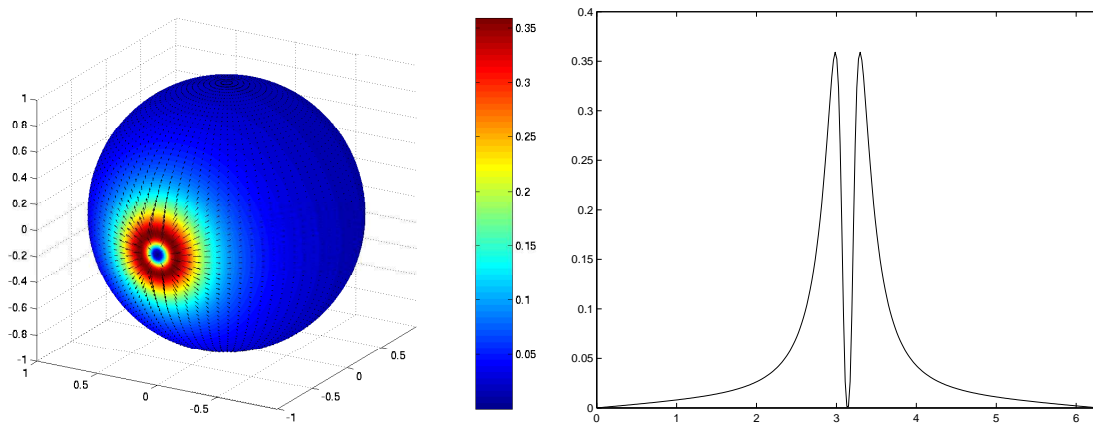


Figure 3: Real part of the vectorial Σ -scaling-function φ_τ^5 for $\tau = 0.125$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

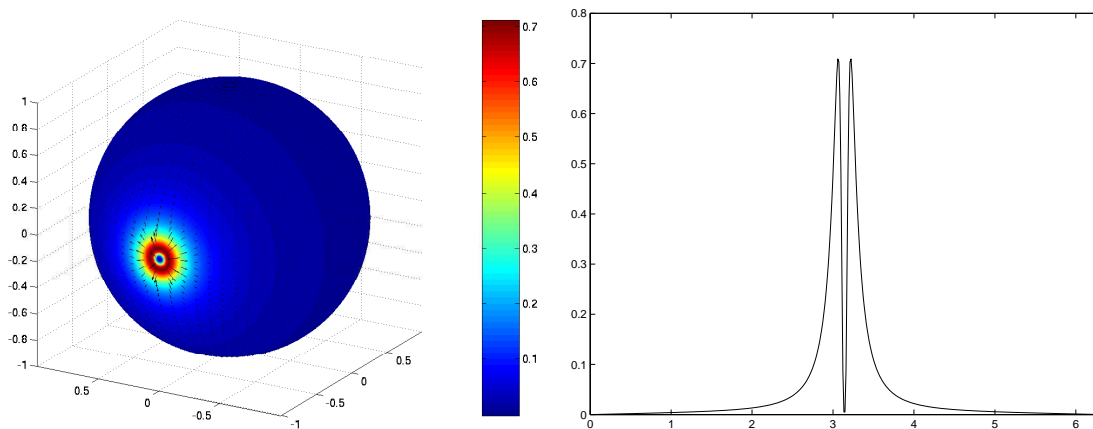


Figure 4: Real part of the vectorial Σ -scaling-function φ_τ^5 for $\tau = 0.0625$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

4.2 Scale Continuous Reconstruction Formula

It is not difficult to see that the wavelet functions Ψ_τ^i , $i \in \{1, \dots, 3\}$, ψ_τ^i , $i \in \{4, \dots, 6\}$ and Ψ_τ^i , $i \in \{7, \dots, 9\}$ behave like $O(\tau^{-1})$, hence, the convergence of the following integrals in the reconstruction theorem is guaranteed.

THEOREM 4.5 *Let f be of class $l_{tan}^2(\Sigma)$. Then the reconstruction formula*

$$\int_0^\infty (WT)^i(f)(\tau, \cdot) \frac{d\tau}{\tau} = \begin{cases} P(0, 0; k)f & , \quad i = 1, 2 \\ 0 & , \quad i = 3, 9 \\ f & , \quad i = 4, 5 \\ P_{curl}(0, 0; k)f & , \quad i = 6 \\ Q(0, 0; k)f & , \quad i = 8, 9 \end{cases}$$

holds in the sense of $\|\cdot\|_{l^2(\Sigma)}$.

Proof. Let $R > 0$ be arbitrary. Let Φ_R^i be a scalar Σ -scaling function of type i , $i = 1, 2, 3$. Then we have by observing Fubini's theorem

$$\Phi_R^i(x, y) = \int_R^\infty \Psi_\tau^i(x, y) \frac{d\tau}{\tau}, \quad (x, y) \in \Sigma \times \Sigma.$$

Thus, we obtain, for $f \in l_{tan}^2(\Sigma)$,

$$\begin{aligned} \int_R^\infty (WT)^i(f)(\tau, \cdot) \frac{d\tau}{\tau} &= \int_R^\infty \int_\Sigma \Psi_\tau^i(\cdot, y) f(y) d\omega(y) \frac{d\tau}{\tau} \\ &= \int_\Sigma \int_R^\infty \Psi_\tau^i(\cdot, y) \frac{d\tau}{\tau} f(y) d\omega(y) \\ &= \int_\Sigma \Phi_R^i(\cdot, y) f(y) d\omega(y). \end{aligned}$$

The limit $R \rightarrow 0$ in connection with Corollary 4.1 yields the desired result.

Now let ϕ_R^i be a vectorial Σ -scaling function of type i , $i = 4, 5, 6$. Then we again have by Fubini's theorem

$$\varphi_R^i(x, y) = \int_R^\infty \psi_\tau^i(x, y) \frac{d\tau}{\tau}, \quad (x, y) \in \Sigma \times \Sigma,$$

and obtain, for $f \in l_{tan}^2(\Sigma)$,

$$\begin{aligned} \int_R^\infty (WT)^i(f)(\tau, \cdot) \frac{d\tau}{\tau} &= \int_R^\infty \nu(\cdot) \wedge \int_\Sigma \psi_\tau^i(\cdot, y) \wedge f(y) d\omega(y) \frac{d\tau}{\tau} \\ &= \nu(\cdot) \wedge \int_\Sigma \left(\int_R^\infty \psi_\tau^i(\cdot, y) \frac{d\tau}{\tau} \right) \wedge f(y) d\omega(y) \\ &= \nu(\cdot) \wedge \int_\Sigma \varphi_R^i(\cdot, y) \wedge f(y) d\omega(y). \end{aligned}$$

The limit $R \rightarrow 0$ in connection with Corollary 4.1 again yields the desired result.

For the case of the tensorial Σ -scaling functions and wavelets, i.e., $i = 7, 8, 9$, the proof can be performed analogously. \square

Note that the properties of the scalar, vectorial and tensorial Σ -wavelets of type i , $i = 1, \dots, 9$, (analogously to variants of spherical wavelets developed in [9], [10]) do not presume the zero-mean property of Ψ_τ^i , ψ_τ^i or Ψ_τ^i . The wavelets constructed in this way, therefore, do not satisfy a substantial condition of the Euclidean concept.

4.3 Spectral Representation

In this section we consider the case $\Sigma = \Omega_\alpha$, i.e., the regular surface is a sphere with radius $\alpha > 0$. Since the vectorial Σ -scaling function φ_τ^5 is most important for constructive approximation of vector fields, we show, that it can be represented in the form

$$\varphi_\tau^5(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\varphi_\tau^5)^\wedge(1; n) \xi P_n(\xi \cdot \eta) + \sum_{n=1}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{\sqrt{n(n+1)}} (\varphi_\tau^5)^\wedge(2; n) \nabla_\xi^* P_n(\xi \cdot \eta)$$

with $x = \alpha\xi$, $y = \alpha\eta$, $\xi, \eta \in \Omega$, where P_n is the Legendre polynomial of degree n and ∇^* is the surface gradient. This is the standard representation in the theory of vectorial product kernels and vectorial multiresolution theory presented in [9].

THEOREM 4.6 *Restricted to a sphere Ω_α the scaling function φ_τ^5 with $k \neq 0$ admits the following representation*

$$\varphi_\tau^5(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\varphi_\tau^5)^\wedge(1; n) \xi P_n(\xi \cdot \eta) + \sum_{n=1}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{\sqrt{n(n+1)}} (\varphi_\tau^5)^\wedge(2; n) \nabla_\xi^* P_n(\xi \cdot \eta)$$

with $x = \alpha\xi$, $y = \alpha\eta$, $\xi, \eta \in \Omega$ and

$$\begin{aligned} (\varphi_\tau^5)^\wedge(1; n) = & \frac{i\pi\alpha^2}{2\sqrt{\alpha}} \left(-\frac{\sqrt{\alpha+\tau}}{2\alpha} J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k(\alpha+\tau)), \right. \\ & + \frac{1}{\sqrt{\alpha+\tau}} k J'_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k(\alpha+\tau)), \\ & + \frac{\sqrt{\alpha-\tau}}{2\alpha} J_{n+1/2}(k(\alpha-\tau)) H_{n+1/2}^{(1)}(k\alpha), \\ & \left. - \frac{1}{\sqrt{\alpha-\tau}} k J_{n+1/2}(k(\alpha-\tau)) H_{n+1/2}^{(1)'}(k\alpha) \right), \quad n \in \mathbb{N}_0, \end{aligned}$$

$$\begin{aligned} (\varphi_\tau^5)^\wedge(2; n) = & \frac{i\pi\alpha^2 \sqrt{n(n+1)}}{2\sqrt{\alpha^3}} \left(\frac{1}{\sqrt{\alpha+\tau}} J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k(\alpha+\tau)), \right. \\ & \left. + \frac{1}{\sqrt{\alpha-\tau}} J_{n+1/2}(k(\alpha-\tau)) H_{n+1/2}^{(1)}(k\alpha) \right), \quad n \in \mathbb{N}, \end{aligned}$$

$\tau \in (0, \alpha)$ and $\alpha > 0$. J_ν and $H_\nu^{(1)}$ are, as usual, the Bessel and Hankel functions of order ν .

Proof. The desired result can be obtained by some easy but lengthy calculations using the identity (see [23])

$$\frac{\exp(ik|x-y|)}{|x-y|} = \frac{i\pi}{2\sqrt{|x||y|}} \sum_{n=0}^{\infty} (2n+1) J_{n+1/2}(k|x|) H_{n+1/2}^{(1)}(k|y|) P_n(\xi \cdot \eta)$$

with $x = |x|\xi$, $y = |y|\eta$, $\xi, \eta \in \Omega$, $|x| < |y|$ and

$$|x + \tau\nu(x)| = \alpha + \tau, \quad x \in \Omega_\alpha, \quad \tau \in (-\alpha, \infty).$$

□

The representation in Theorem 4.6 is the standard spectral representation for product kernels as used in [9]. It is worth mentioning, that the restriction $\tau < \alpha$ does not really matter since we are finally interested in the limit $\tau \rightarrow 0$.

LEMMA 4.7 *For the symbols $(\varphi_\tau^5)^\wedge(1; n)$, $n \in \mathbb{N}_0$, and $(\varphi_\tau^5)^\wedge(2; n)$, $n \in \mathbb{N}$, of the spherical vectorial scaling function φ_τ^5 we obtain*

$$\begin{aligned} \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} (\varphi_\tau^5)^\wedge(1; n) &= -1 \quad \text{for all } n \in \mathbb{N}_0, \\ \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} (\varphi_\tau^5)^\wedge(2; n) &= 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Proof. Taking the limit $\tau \rightarrow 0$ for $(\varphi_\tau^5)^\wedge(1; n)$ we find

$$\begin{aligned} \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} (\varphi_\tau^5)^\wedge(1; n) &= \frac{i\pi\alpha^2}{2\sqrt{\alpha}} \left(-\frac{\sqrt{\alpha}}{2\alpha} J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k\alpha) + \frac{1}{\sqrt{\alpha}} k J'_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k\alpha), \right. \\ &\quad \left. + \frac{\sqrt{\alpha}}{2\alpha} J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k\alpha) - \frac{1}{\sqrt{\alpha}} k J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k\alpha) \right) \\ &= \frac{i\pi k\alpha}{2} \left(J'_{n+1/2}(k\alpha) H_{n+1/2}^{(1)}(k\alpha) - J_{n+1/2}(k\alpha) H_{n+1/2}^{(1)'}(k\alpha) \right) \end{aligned}$$

for $n \in \mathbb{N}_0$. Using the Wronskian of two Bessel functions we get

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} (\varphi_\tau^5)^\wedge(1; n) = \frac{i\pi k\alpha}{2} \left(\frac{2i}{\pi k\alpha} \right) = -1$$

which is the desired result for $(\varphi_\tau^5)^\wedge(1; n)$, $n \in \mathbb{N}_0$. The result for the symbol $(\varphi_\tau^5)^\wedge(2; n)$ immediately follows from the continuity of the Bessel and Hankel functions. \square

The above lemma can be used to prove a kind of reassurance of the jump relation vor the potential $P_{curl}f$ in the spectral sense. For this proof, let $f \in c_{tan}^{(0)}(\Omega_\alpha)$ be given. Observing the Helmholtz decomposition theorem for continuous spherical vector fields (see [9], there exist scalar fields $F_2, F_3 \in C^{(0)}(\Omega_\alpha)$ such that

$$f(x) = (\nabla^* F_2)(x) + (L^* F_3)(x), \quad x \in \Omega_\alpha,$$

where ∇^* is the surface gradient and L^* is the surface curl gradient (see [9]).

Thus, we get, with $x = \alpha\xi$, $y = \alpha\eta$, $\xi, \eta \in \Omega$

$$\begin{aligned} &\int_{\Omega_\alpha} \varphi_\tau^5(x, y) \wedge f(y) d\omega(y) \\ &= \int_{\Omega_\alpha} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\varphi_\tau^5)^\wedge(1; n) \xi P_n(\xi \cdot \eta) \wedge ((\nabla^* F_2)(y) + (L^* F_3)(y)) d\omega(y) \\ &\quad + \int_{\Omega_\alpha} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{\sqrt{n(n+1)}} (\varphi_\tau^5)^\wedge(2; n) \nabla_\xi^* P_n(\xi \cdot \eta) \wedge ((\nabla^* F_2)(y) + (L^* F_3)(y)) d\omega(y) \\ &= \int_{\Omega_\alpha} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\varphi_\tau^5)^\wedge(1; n) P_n(\xi \cdot \eta) ((L^* F_2)(y) - (\nabla^* F_3)(y)) d\omega(y) \\ &\quad + \int_{\Omega_\alpha} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{\sqrt{n(n+1)}} (\varphi_\tau^5)^\wedge(2; n) \nabla_\xi^* P_n(\xi \cdot \eta) \wedge ((\nabla^* F_2)(y) + (L^* F_3)(y)) d\omega(y) \end{aligned} \quad (8)$$

Now, the scalar kernel

$$\Phi_\tau^5(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\varphi_\tau^5)^\wedge(1; n) P_n(\xi \cdot \eta), \quad x, y \in \Omega_\alpha$$

can be interpreted as a scalar scaling function which fulfills

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Omega_\alpha} \Phi_\tau^5(x, y) f(y) d\omega(y) = -f(x), \quad x \in \Omega_\alpha,$$

for all $f \in c_{tan}^{(0)}(\Omega_\alpha)$. Since the symbol $(\varphi_\tau^5)^\wedge(2; n)$ tends to 0 for τ tending to 0 the second summand in (8) vanishes in the limit $\tau \rightarrow 0$. Thus we obtain

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Omega_\alpha} \varphi_\tau^5(x, y) \wedge f(y) d\omega(y) = -(L^* F_2)(x) + (\nabla^* F_3)(x), \quad x \in \Omega_\alpha.$$

Finally we get

$$\begin{aligned}
\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \nu(x) \wedge \int_{\Omega_\alpha} \varphi_\tau^5(x, y) \wedge f(y) d\omega(y) &= \nu(x) \wedge \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Omega_\alpha} \varphi_\tau^5(x, y) \wedge f(y) d\omega(y) \\
&= \xi \wedge \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Omega_\alpha} \varphi_\tau^5(x, y) \wedge f(y) d\omega(y) \\
&= \xi \wedge (-(L^*F_2)(x) + (\nabla^*F_3)(x)) \\
&= (\nabla^*F_2)(x) + (L^*F_3)(x) \\
&= f(x), \quad x \in \Omega_\alpha,
\end{aligned}$$

which is the desired result.

4.4 Scale Discretized Reconstruction Formula

Until now we were concerned with a scale continuous approach to wavelets. In what follows, scale discrete Σ -scaling functions and wavelets of type i will be introduced. We start with the choice of a sequence which divides the continuous scale interval $(0, \infty)$ into discrete pieces. More explicitly, $(\tau_j)_{j \in \mathbb{Z}}$ denotes a sequence of real numbers satisfying

$$\lim_{j \rightarrow \infty} \tau_j = 0 \quad \lim_{j \rightarrow -\infty} \tau_j = \infty.$$

For example, one may choose $\tau_j = 2^{-j}$, $j \in \mathbb{Z}$ (note that in this case, $2\tau_{j+1} = \tau_j$, $j \in \mathbb{Z}$).

Given a scalar Σ -scaling function $\{\Phi_\tau^i\}_{\tau > 0}$ of type i , $i = 1, 2, 3$, then we clearly define the (scale) *discretized scalar Σ -scaling functions* of type i by $\{\Phi_{\tau_j}^i\}_{j \in \mathbb{Z}}$. Analogously we proceed with the vectorial and tensorial Σ -scaling functions and get (scale) *discretized vectorial Σ -scaling functions* of type i by $\{\varphi_{\tau_j}^i\}_{j \in \mathbb{Z}}$, $i = 4, 5, 6$, and (scale) *discretized tensorial Σ -scaling functions* of type i by $\{\Phi_{\tau_j}^i\}_{j \in \mathbb{Z}}$, $i = 7, 8, 9$. In doing so, by Corollary 4.1, we immediately get the following result.

THEOREM 4.8 For $f \in l_{tan}^2(\Sigma)$

$$\begin{aligned}
\lim_{j \rightarrow \infty} \int_{\Sigma} \Phi_{\tau_j}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} P(0, 0; k)f & , \quad i = 1, 2 \\ 0 & , \quad i = 3, \end{cases} \\
\lim_{j \rightarrow \infty} \nu(\cdot) \wedge \int_{\Sigma} \varphi_{\tau_j}^i(\cdot, y) \wedge f(y) d\omega(y) &= \begin{cases} f & , \quad i = 4, 5 \\ P_{curl}(0, 0; k)f & , \quad i = 6, \end{cases} \\
\lim_{j \rightarrow \infty} \nu(\cdot) \wedge \int_{\Sigma} \Phi_{\tau_j}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} Q(0, 0; k)f & , \quad i = 7, 8, \\ 0 & , \quad i = 9 \end{cases}
\end{aligned}$$

holds in the $\|\cdot\|_{l^2(\Sigma)}$ -sense.

Our procedure canonically leads us to the following type of scale discretized wavelets.

DEFINITION 4.9 Let $\{\Psi_\tau^i\}_{\tau > 0}$, $i = 1, 2, 3$, $\{\psi_\tau^i\}_{\tau > 0}$, $i = 4, 5, 6$ and $\{\Phi_\tau^i\}_{\tau > 0}$, $i = 7, 8, 9$ be the scalar, vectorial and tensorial Σ -scaling functions of type i . Then the (scale) *discretized scalar Σ -wavelet function* of type i , $\Psi_{\tau_j}^i$, $i = 1, 2, 3$, the *discretized vectorial Σ -wavelet function* of type i , $\psi_{\tau_j}^i$, $i = 4, 5, 6$, and the *discretized tensorial Σ -wavelet function* of type i , $\Psi_{\tau_j}^i$, $i = 7, 8, 9$, are defined by

$$\begin{aligned}
\Psi_{\tau_j}^i(\cdot, \cdot) &= \int_{\tau_{j+1}}^{\tau_j} \Psi_\tau^i(\cdot, \cdot) \frac{d\tau}{\tau}, \quad j \in \mathbb{Z}, \quad i = 1, 2, 3, \\
\psi_{\tau_j}^i(\cdot, \cdot) &= \int_{\tau_{j+1}}^{\tau_j} \psi_\tau^i(\cdot, \cdot) \frac{d\tau}{\tau}, \quad j \in \mathbb{Z}, \quad i = 4, 5, 6, \\
\Psi_{\tau_j}^i(\cdot, \cdot) &= \int_{\tau_{j+1}}^{\tau_j} \Psi_\tau^i(\cdot, \cdot) \frac{d\tau}{\tau}, \quad j \in \mathbb{Z}, \quad i = 7, 8, 9.
\end{aligned}$$

In connection with (5), (6) and (7) it follows that

$$\Psi_{\tau_j}^i(\cdot, \cdot) = - \int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \Phi_{\tau}^i(\cdot, \cdot) \frac{d\tau}{\tau} = \Phi_{\tau_{j+1}}^i(\cdot, \cdot) - \Phi_{\tau_j}^i(\cdot, \cdot), \quad i = 1, 2, 3, \quad (9)$$

$$\psi_{\tau_j}^i(\cdot, \cdot) = - \int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \varphi_{\tau}^i(\cdot, \cdot) \frac{d\tau}{\tau} = \varphi_{\tau_{j+1}}^i(\cdot, \cdot) - \varphi_{\tau_j}^i(\cdot, \cdot), \quad i = 4, 5, 6, \quad (10)$$

$$\Psi_{\tau_j}^i(\cdot, \cdot) = - \int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \Phi_{\tau}^i(\cdot, \cdot) \frac{d\tau}{\tau} = \Phi_{\tau_{j+1}}^i(\cdot, \cdot) - \Phi_{\tau_j}^i(\cdot, \cdot), \quad i = 7, 8, 9. \quad (11)$$

The formulas (9), (10) and (11) are called scalar, vectorial and tensorial *(scale) discretized Σ -scaling equation of type i* .

Assume now that f is a function of class $l_{tan}^2(\Sigma)$. Observing the discretized Σ -scaling equations of type i we get, for $J \in \mathbb{Z}$ and $N \in \mathbb{N}$,

$$\begin{aligned} \int_{\Sigma} \Phi_{\tau_{J+N}}^i(\cdot, y) f(y) d\omega(y) &= \int_{\Sigma} \Phi_{\tau_J}^i(\cdot, y) f(y) d\omega(y) + \sum_{j=J}^{J+N-1} \int_{\Sigma} \Psi_{\tau_j}^i(\cdot, y) f(y) d\omega(y), \quad i = 1, 2, 3, \\ \nu(\cdot) \wedge \int_{\Sigma} \varphi_{\tau_{J+N}}^i(\cdot, y) \wedge f(y) d\omega(y) &= \nu(\cdot) \wedge \int_{\Sigma} \varphi_{\tau_J}^i(\cdot, y) \wedge f(y) d\omega(y) \\ &\quad + \sum_{j=J}^{J+N-1} \nu(\cdot) \wedge \int_{\Sigma} \psi_{\tau_j}^i(\cdot, y) \wedge f(y) d\omega(y), \quad i = 4, 5, 6, \\ \nu(\cdot) \wedge \int_{\Sigma} \Phi_{\tau_{J+N}}^i(\cdot, y) f(y) d\omega(y) &= \nu(\cdot) \wedge \int_{\Sigma} \Phi_{\tau_J}^i(\cdot, y) f(y) d\omega(y) \\ &\quad + \sum_{j=J}^{J+N-1} \nu(\cdot) \wedge \int_{\Sigma} \Psi_{\tau_j}^i(\cdot, y) f(y) d\omega(y), \quad i = 7, 8, 9. \end{aligned}$$

Therefore we are able to formulate the following corollary.

COROLLARY 4.10 *Let $\{\Psi_{\tau_j}^i\}_{j \in \mathbb{Z}}$, $i = 1, 2, 3$, $\{\psi_{\tau_j}^i\}_{j \in \mathbb{Z}}$, $i = 4, 5, 6$ and $\{\Psi_{\tau_j}^i\}_{j \in \mathbb{Z}}$, $i = 7, 8, 9$ be the (scale) discretized scalar, vectorial and tensorial Σ -scaling functions of type i . Then the multiscale representation of a function $f \in l_{tan}^2(\Sigma)$ and its potentials*

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} \int_{\Sigma} \Psi_{\tau_j}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} P(0, 0; k) f & , \quad i = 1, 2, \\ 0 & , \quad i = 3, \end{cases} \\ \sum_{j=-\infty}^{+\infty} \nu(\cdot) \wedge \int_{\Sigma} \psi_{\tau_j}^i(\cdot, y) \wedge f(y) d\omega(y) &= \begin{cases} f & , \quad i = 4, 5, \\ P_{curl}(0, 0; k) f & , \quad i = 6, \end{cases} \\ \sum_{j=-\infty}^{+\infty} \nu(\cdot) \wedge \int_{\Sigma} \Psi_{\tau_j}^i(\cdot, y) f(y) d\omega(y) &= \begin{cases} Q(0, 0; k) f & , \quad i = 7, 8, \\ 0 & , \quad i = 9 \end{cases} \end{aligned}$$

holds in the $\|\cdot\|_{l^2(\Sigma)}$ -sense.

Corollary 4.10 admits the following reformulation.

COROLLARY 4.11 *Under the assumption of Corollary 4.10*

$$P_{\tau_J}^i(f) + \sum_{j=-\infty}^J (WT)^i(f)(\tau_j; \cdot) = \begin{cases} P(0, 0; k) f & , \quad i = 1, 2, \\ 0 & , \quad i = 3, 9, \\ f & , \quad i = 4, 5, \\ P_{curl}(0, 0; k) f & , \quad i = 6, \\ Q(0, 0; k) f & , \quad i = 7, 8, \end{cases}$$

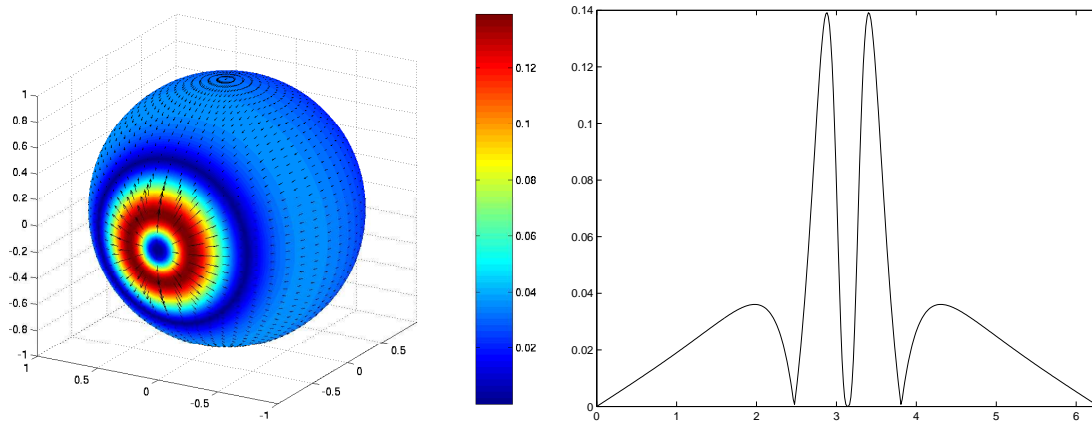


Figure 5: Real part of the discrete vectorial Σ -wavelet-function φ_j^5 for $j = 1$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

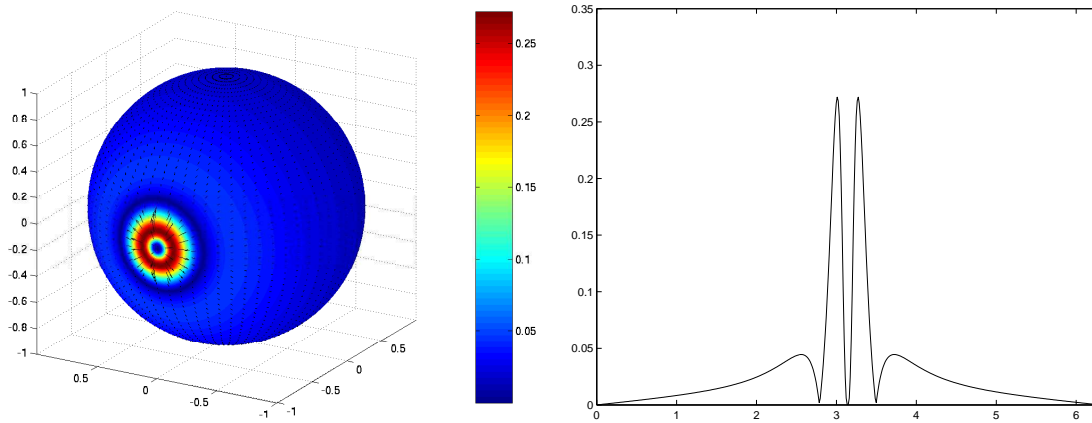


Figure 6: Real part of the discrete vectorial Σ -wavelet-function φ_j^5 for $j = 2$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

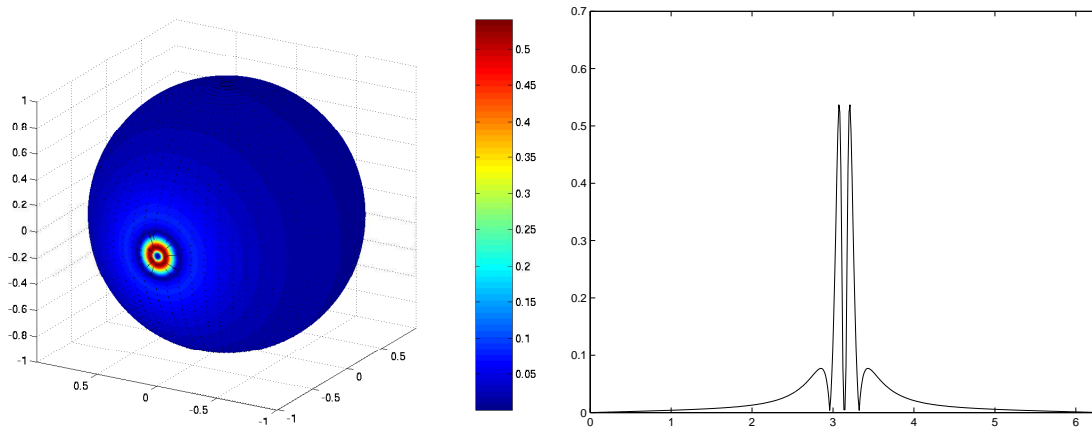


Figure 7: Real part of the discrete vectorial Σ -wavelet-function φ_j^5 for $j = 3$ and $k = 1$ on the unit sphere $\Sigma = \Omega$ (left) and a sectional illustration of its absolute value along the equator (right)

for every $J \in \mathbb{Z}$ (in the sense of the $\|\cdot\|_{l^2(\Sigma)}$ -norm), where $P_{\tau_J}^i(f)$ is given by

$$P_{\tau_J}^i(F)(x) = \begin{cases} \int_{\Sigma} \Phi_{\tau_J}^i(x, y) f(y) d\omega(y) & , \quad i = 1, 2, 3, \\ \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_J}^i(x, y) \wedge f(y) d\omega(y) & , \quad i = 4, 5, 6, \\ \nu(x) \wedge \int_{\Sigma} \Phi_{\tau_J}^i(x, y) f(y) d\omega(y) & , \quad i = 7, 8, 9. \end{cases} \quad x \in \Sigma,$$

and the (scale) discretized Σ -wavelet transform of type i is defined by

$$(WT)^i : l^2(\Sigma) \mapsto \left\{ h : \mathbb{Z} \times \Sigma \rightarrow \mathbb{R}^3 \mid \sum_{j=-\infty}^{\infty} \int_{\Sigma} |h(j; y)|^2 d\omega(y) < \infty \right\}$$

with

$$(WT)^i(F)(j; x) = \begin{cases} \int_{\Sigma} \Psi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 1, 2, 3 \\ \nu(x) \wedge \int_{\Sigma} \psi_{\tau_j}^i(x, y) \wedge f(y) d\omega(y) & , \quad i = 4, 5, 6 \\ \nu(x) \wedge \int_{\Sigma} \Psi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 7, 8, 9. \end{cases} \quad x \in \Sigma.$$

4.5 Scale and Detail Spaces

Comparing this result with the continuous analogue (4.5) we notice that the subdivision of the continuous scale interval $(0, \infty)$ into discrete pieces means substitution of the integral over τ by an associated discrete sum.

As in the spherical theory of wavelets (see [7], [8]), the operators $R_{\tau_j}^i, P_{\tau_j}^i$ defined by

$$\begin{aligned} R_{\tau_j}^i(f) &= \begin{cases} \int_{\Sigma} \Psi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 1, 2, 3 \\ \nu(x) \wedge \int_{\Sigma} \psi_{\tau_j}^i(x, y) \wedge f(y) d\omega(y) & , \quad i = 4, 5, 6 \\ \nu(x) \wedge \int_{\Sigma} \Psi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 7, 8, 9. \end{cases} \\ P_{\tau_j}^i(f) &= \begin{cases} \int_{\Sigma} \Phi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 1, 2, 3 \\ \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_j}^i(x, y) \wedge f(y) d\omega(y) & , \quad i = 4, 5, 6 \\ \nu(x) \wedge \int_{\Sigma} \Phi_{\tau_j}^i(x, y) f(y) d\omega(y) & , \quad i = 7, 8, 9. \end{cases} \end{aligned}$$

for $f \in l_{tan}^2(\Sigma)$, may be understood as band pass and low pass filter, respectively. The *scale spaces* $\mathcal{V}_{\tau_j}^i$ and the *detail spaces* $\mathcal{W}_{\tau_j}^i$ of type i are defined by

$$\begin{aligned} \mathcal{V}_{\tau_j}^i &= P_{\tau_j}^i(l_{tan}^2(\Sigma)) = \left\{ P_{\tau_j}^i(f) \mid f \in l_{tan}^2(\Sigma) \right\}, \\ \mathcal{W}_{\tau_j}^i &= R_{\tau_j}^i(l_{tan}^2(\Sigma)) = \left\{ R_{\tau_j}^i(f) \mid f \in l_{tan}^2(\Sigma) \right\}, \end{aligned}$$

respectively. From the identity

$$P_{\tau_{j+1}}^i(F) = P_{\tau_j}^i(F) + R_{\tau_j}^i(F)$$

for all $J \in \mathbb{Z}$, it easily follows that

$$\mathcal{V}_{\tau_{j+1}}^i = \mathcal{V}_{\tau_j}^i + \mathcal{W}_{\tau_j}^i. \quad (12)$$

However, it should be remarked that the sum (12) generally is neither direct nor orthogonal. The equation (12) may be interpreted in the following way: The set $\mathcal{V}_{\tau_j}^i$ contains a $P_{\tau_j}^i$ -filtered version of a function belonging to the class $l^2(\Sigma)$. The lower the scale, the stronger the intensity of filtering. By adding ' $R_{\tau_j}^i$ -details' contained in the space $\mathcal{W}_{\tau_j}^i$ the space $\mathcal{V}_{\tau_{j+1}}^i$ is created, which consists of a filtered version at resolution $j+1$. Obviously, for $i = 4, 5$,

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}_{\tau_j}^i}^{\|\cdot\|_{l^2(\Sigma)}} = l_{tan}^2(\Sigma).$$

Moreover,

$$\begin{aligned} \overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}_{\tau_j}^i}^{\|\cdot\|_{l^2(\Sigma)}} &= P(0, 0; k)(l_{tan}^2(\Sigma)), \quad i = 1, 2, \\ \overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}_{\tau_j}^6}^{\|\cdot\|_{l^2(\Sigma)}} &= P_{curl}(0, 0; k)(l_{tan}^2(\Sigma)), \\ \overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}_{\tau_j}^i}^{\|\cdot\|_{l^2(\Sigma)}} &= Q(0, 0; k)(l_{tan}^2(\Sigma)), \quad i = 7, 8. \end{aligned}$$

4.6 A Tree Algorithm

Next we deal with some aspects of scientific computing (for a similar approach in spherical theory see [11]). We are interested in a pyramid scheme for the (approximate) recursive computation of the integrals $P_{\tau_j}(f)$ and $R_{\tau_j}(f)$ starting from an initial approximation of a given function $f \in l_{tan}^2(\Sigma)$. The tree algorithm (pyramid scheme) is based on the existence of a 'reproducing kernel function' on the regular surface Σ . A pyramid scheme is a tree algorithm with the following ingredients. Starting from a sufficiently large $J \in \mathbb{N}$ such that for all $x \in \Sigma$

$$P_{\tau_J}^5(f) \simeq \nu(x) \wedge \sum_{k=1}^{N_J} \varphi_{\tau_J}^5(x, y_k^{N_J}) \wedge a_k^{N_J} \simeq f \quad (13)$$

we want to calculate vectorial coefficients

$$a^{N_j} \in \mathbb{R}^{N_j} \times \mathbb{R}^3, a^{N_j} = \left(a_1^{N_j}, \dots, a_{N_j}^{N_j} \right)^T, \quad j = J_0, \dots, J,$$

such that the following statements hold true:

1. The vectors a^{N_j} , $j = J_0, \dots, J-1$, are obtainable by recursion starting from the vector a^{N_J} .
2. For $j = J_0, \dots, J$

$$P_{\tau_j}^5(f)(x) = \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_j}^5(x, y) \wedge f(y) d\omega(y) \simeq \nu(x) \wedge \sum_{k=1}^{N_j} \varphi_{\tau_j} \left(x, y_k^{N_j} \right) \wedge a_k^{N_j}.$$

For $j = J_0 + 1, \dots, J$

$$R_{\tau_{j-1}}^5(f)(x) = \nu(x) \wedge \int_{\Sigma} \psi_{\tau_{j-1}}^5(x, y) \wedge f(y) d\omega(y) \simeq \sum_{k=1}^{N_{j-1}} \psi_{\tau_{j-1}} \left(x, y_k^{N_{j-1}} \right) \wedge a_k^{N_{j-1}}.$$

(the symbol ' \simeq ' always means that the error is assumed to be negligible).

In the scheme we base the numerical integration on certain approximate formulae associated to known weights $w_k^{N_j} \in \mathbb{R}$ and prescribed knots $y_k^{N_j} \in \Sigma$, $j = J_0, \dots, J$. This may be established, for example, by transforming the integrals over the regular surface Σ to integrals over the (unit) sphere Ω in case an explicit transformation $\Theta : \Omega \rightarrow \Sigma$ is given (see [6], [22]). Note that j denotes the scale of the discretized scaling function, N_j is the number of integration points to the accompanying scale j , and k denotes the index of the integration knot within the integration formulae under consideration, i.e.,

$$P_j^5(f)(x) \simeq \nu(x) \wedge \sum_{k=1}^{N_j} w_k^{N_j} \varphi_j^5(x, y_k^{N_j}) \wedge f(y_k^{N_j}),$$

$$j = J_0, \dots, J,$$

$$R_{j-1}^5(f)(x) \simeq \nu(x) \wedge \sum_{k=1}^{N_{j-1}} w_k^{N_{j-1}} \psi_{j-1}^5(x, y_k^{N_{j-1}}) \wedge f(y_k^{N_{j-1}}),$$

$$j = J_0 + 1, \dots, J.$$

The pyramid scheme – as every recursive implementation – is divided into two parts, the initial step and the recursion step, here called the pyramid step.

Initial Step. For a suitable large integer J , $P_{\tau_J}^5(f)(x)$ is sufficiently close to the right hand side of (13) for all $x \in \Sigma$. Thus we simply get by (4.6)

$$a_k^{N_J} = w_k^{N_J} f(y_k^{N_J}), \quad k = 1, \dots, N_J.$$

Pyramid Step. The essential idea for the development of our recursive scheme is the existence of a (symmetric) kernel function $\Xi_j : \Sigma \times \Sigma \rightarrow \mathbb{R}$ such that

$$\varphi_{\tau_j}^5(x, y) \simeq \int_{\Sigma} \varphi_{\tau_j}^5(z, x) \Xi_j(y, z) d\omega(z) \quad (14)$$

and

$$\Xi_j(x, y) \simeq \int_{\Sigma} \Xi_j(z, x) \Xi_{j+1}(y, z) d\omega(z) \quad (15)$$

for $j = J_0, \dots, J$.

Since our scaling functions are non-bandlimited, the scale spaces \mathcal{V}_j are infinite dimensional. This leads us to choose the functions Ξ_j to be an approximation of the scalar delta kernel on the regular surface Σ . Such multiscale scalar approximations have been presented in a similar context as in this article in [12] for the scalar Laplace equation or in [13] for the scalar Helmholtz equation. Both results can be used for Ξ_j at this point because they both establish a scalar multiscale approximation on a regular surface Σ . By virtue of the approximate integration rules we thus get

$$\begin{aligned} \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_j}^5(\cdot, y) \wedge f(y) d\omega(y) &\simeq \int_{\Sigma} \Xi_j(y, z) \left(\nu(x) \wedge \int_{\Sigma} \varphi_{\tau_j}^5(\cdot, z) \wedge f(y) d\omega(z) \right) d\omega(y) \\ &\simeq \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_j}^5(\cdot, z) \wedge \left(\int_{\Sigma} \Xi_j(y, z) f(y) d\omega(y) \right) d\omega(z) \\ &\simeq \nu(x) \wedge \sum_{k=1}^{N_j} \varphi_{\tau_j}^5(\cdot, y_k^{N_j}) \wedge a_k^{N_j} \end{aligned}$$

for $j = J_0, \dots, J-1$, where we have set

$$a_k^{N_j} = w_k^{N_j} \int_{\Sigma} \Xi_j(y_k^{N_j}, y) f(y) d\omega(y)$$

for $j = J_0, \dots, J-1$ and $k = 1, \dots, N_j$. Hence, in connection with (15), we find

$$\begin{aligned}
a_k^{N_j} &= w_k^{N_j} \int_{\Sigma} \Xi_j(y_k^{N_j}, y) f(y) d\omega(y) \\
&\simeq w_k^{N_j} \int_{\Sigma} \int_{\Sigma} \Xi_{j+1}(z, y) \Xi_j(y_k^{N_j}, z) d\omega(z) f(y) d\omega(y) \\
&\simeq w_k^{N_j} \int_{\Sigma} \Xi_j(y_k^{N_j}, z) \left(\int_{\Sigma} \Xi_{j+1}(z, y) f(y) d\omega(y) \right) d\omega(z) \\
&\simeq w_k^{N_j} \sum_{l=1}^{N_{j+1}} w_l^{N_{j+1}} \Xi_j(y_k^{N_j}, y_l^{N_{j+1}}) \int_{\Sigma} \Xi_{j+1}(y_l^{N_{j+1}}, y) f(y) d\omega(y) \\
&= w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j(y_k^{N_j}, y_l^{N_{j+1}}) a_l^{N_{j+1}}.
\end{aligned}$$

for $j = J-1, \dots, J_0$ and $k = 1, \dots, N_j$.

We see that the coefficients $a_k^{N_{j-1}}$ can be calculated recursively from $a_l^{N_j}$ for the initial level $J, a_k^{N_{J-2}}$ can be deduced from $a_l^{N_{J-1}}$, etc. Finally, we get as a reconstruction scheme

$$\begin{aligned}
P_{\tau_j}^5(f) &\simeq \nu(x) \wedge \sum_{k=1}^{N_j} \varphi_{\tau_j}^5(\cdot, y_k^{N_j}) \wedge a_k^{N_j}, \quad j = J_0, \dots, J, \\
R_{\tau_{j-1}}^5(f) &\simeq \nu(x) \wedge \sum_{k=1}^{N_{j-1}} \psi_{\tau_{j-1}}^5(\cdot, y_k^{N_{j-1}}) \wedge a_k^{N_{j-1}}, \quad j = J_0 + 1, \dots, J.
\end{aligned}$$

Note that the functions $\Xi_j, j = J_0, \dots, J-1$, can be chosen independently of the scaling function $\{\varphi_{\tau_j}^5\}_{j \in \mathbb{Z}}$ used in (16) and (16).

<p>Initial step: For J sufficiently large</p> $a_k^{N_J} = w_k^{N_J} f(y_k^{N_J}), \quad k = 1, \dots, N_J$ <p>Pyramid step: For $j = J_0, \dots, J-1$ and $k = 1, \dots, N_j$</p> $a_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi_j(y_k^{N_j}, y_l^{N_{j+1}}) a_l^{N_{j+1}}$
--

Table 1: Pyramid Scheme (Tree Algorithm)

In conclusion, the above considerations lead us to the following decomposition and reconstruction scheme:

$$\begin{array}{ccccccc}
F & \rightarrow & a^{N_J} & \rightarrow & a^{N_{J-1}} & \rightarrow & \dots \rightarrow a^{N_{J_0+1}} \rightarrow a^{N_{J_0}} \\
& & \downarrow & & \downarrow & & \downarrow & \downarrow \\
& & P_{\tau_J}^5(f) & & P_{\tau_{J-1}}^5(f) & & P_{\tau_{J_0+1}}^5(f) & P_{\tau_{J_0}}^5(f)
\end{array}$$

(decomposition scheme)

$$\begin{array}{ccccccc}
a^{N_{J_0}} & & a^{N_{J_0+1}} & & a^{N_{J-1}} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_{\tau_{J_0}}^5(f) & & P_{\tau_{J_0+1}}^5(f) & & P_{\tau_{J-1}}^5(f) & & \\
\searrow & & \searrow & & \searrow & & \\
R_{\tau_{J_0}}^5(f) & \rightarrow + \rightarrow & R_{\tau_{J_0+1}}^5(f) & \rightarrow + \dots + \rightarrow & R_{\tau_{J-1}}^5(f) & \rightarrow + \rightarrow & P_{\tau_J}^5(f)
\end{array}$$

(reconstruction scheme).

The numerical effort of a pyramid step can drastically be reduced by use of a panel-clustering method (e.g., fast multipole procedures as developed by [15]). In doing so, the evaluations take advantage of the localizing structure of the kernels Ξ_j . Roughly spoken, the kernel is split into a near field and a far field component. The far field component is approximated by a certain expression obtaining the 'low frequency contributions'. For the points close to the evaluation position the evaluation uses the exact near field of the kernel. For the remaining points, the approximate far field contributions are put together.

5 Examples

The first example which we want to present, is a multiscale decomposition of the electromagnetic field of a single point source on an ellipsoidal regular surface given by

$$E(a, b, c) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \right\}$$

where we have chosen $a = 1$, $b = 1.5$ and $c = 2.0$. The field of the single point source located at the point $y \in \mathbb{R}^3$ is given by

$$f(x) = \nu(x) \wedge \nabla_x G(x, y) = \nu(x) \wedge (x - y) \frac{\exp(ik|x - y|)}{|x - y|^2} \left(ik - \frac{1}{|x - y|} \right).$$

Here we have set $y \in \mathbb{R}^3$ to be located inside the ellipsoid given by $y = 0.1(0, 0.5, \sqrt{3}/2)$. The wave number k has been set to be $k = 1.0$. Figure 8 shows a multiscale decomposition of the real part of this complex-valued vector field on the given ellipsoid.

τ_J	E(1.0,1.5,2.0)			
	$\frac{\ P_{\tau_J} f - f\ _{c(\Sigma)}}{\ f\ _{c(\Sigma)}}$	Ratio	$\frac{\ P_{\tau_J} f - f\ _{l^2(\Sigma)}}{\ f\ _{l^2(\Sigma)}}$	Ratio
2^{-1}	0.47420		3.846E-2	
2^{-2}	0.30315	1.56	1.389E-2	2.77
2^{-3}	0.17493	1.73	4.420E-3	3.14
2^{-4}	0.09459	1.85	1.268E-3	3.49
2^{-5}	0.04939	1.91	3.222E-4	3.94
2^{-6}	0.02609	1.89	8.912E-5	3.61
2^{-7}	0.01284	2.03	2.272E-6	3.92

Table 2: Relative error level of a multiscale decomposition at different scales of a single point source

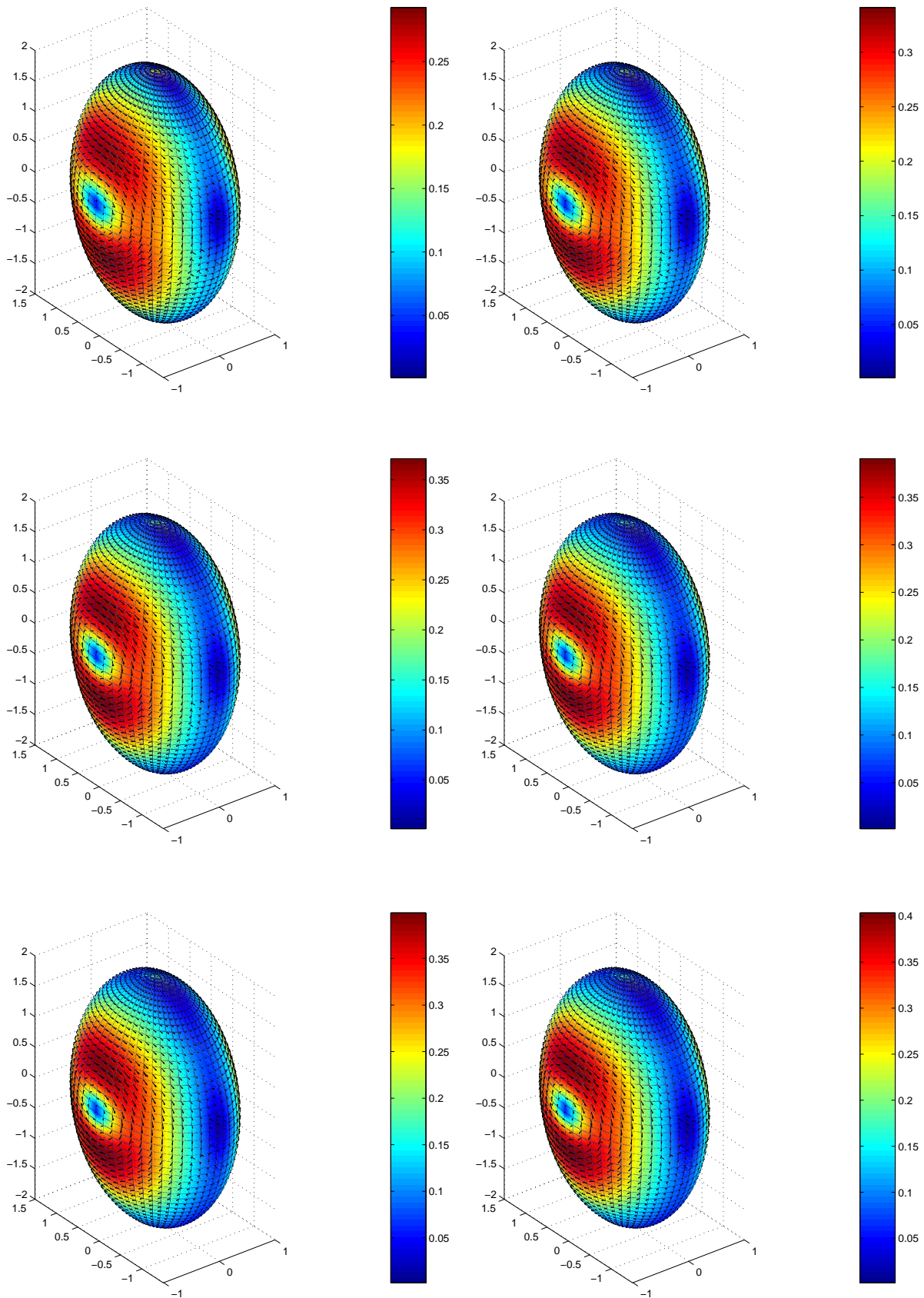


Figure 8: Discrete multiscale decomposition of a single point source on an ellipsoid starting at scale $J = 2$ (upper left) to scale $J = 7$ (lower right)

The second surface we want to discuss is given by an explicit transformation $\Phi : \Omega \rightarrow \Sigma$ from the unit sphere Ω to the regular surface Σ ,

$$\Phi(\xi) = \rho(\xi_1, \xi_2, \xi_3) \begin{pmatrix} a\xi_1 \\ b\xi_2 \\ c\xi_3 \end{pmatrix}, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \Omega \quad (16)$$

with

$$\rho(\xi_1, \xi_2, \xi_3) = d(\xi_1^2 + g\xi_1^3) + e(\xi_2^2 + g\xi_2^3) + f(\xi_3^2 + g\xi_3^3)$$

and the parameters given by $a = 1$, $b = 1.5$, $c = 2$, $d = 1$, $e = 0.7$, $f = 3$ and $g = 0.3$. This regular surface has also been discussed in [2] in the scope of numerical integration on a regular surface. It is designed to demonstrate that the algorithms of this paper do not depend on any symmetry of the regular surface, as the surface defined by (16) has no obvious geometric symmetry.

The function we decompose using our multiscale technique is the trace of an incoming electromagnetic wave on the surface Σ given by

$$f(x) = (d \wedge p) \exp(ikd \cdot x), \quad x \in \Sigma$$

with incoming direction $d = (1, 1, 1)$ and polarization $p = (1, 0, 0)$. The wave number k has been set to be $k = 1.0$. Figure 9 shows a multiscale decomposition of the real part of this complex-valued vector field on the given regular surface. First the scale reconstruction at scale $J = 2$ is presented. To obtain higher scale reconstructions the corresponding detail reconstructions at the following scales have to be added. These detail reconstructions from scale $J = 2$ to scale $J = 5$ are presented in the following figures. The last image in the lower right corner shows the final scale reconstruction at the highest scale $J = 6$.

τ_J	Σ			
	$\frac{\ P_{\tau_J} f - f\ _{c(\Sigma)}}{\ f\ _{c(\Sigma)}}$	Ratio	$\frac{\ P_{\tau_J} f - f\ _{l^2(\Sigma)}}{\ f\ _{l^2(\Sigma)}}$	Ratio
2^{-1}	0.5168		0.2155	
2^{-2}	0.3108	1.66	0.0727	2.96
2^{-3}	0.2106	1.47	0.0200	3.63
2^{-4}	0.1493	1.41	0.0055	3.63
2^{-5}	0.1103	1.35	0.0027	2.03
2^{-6}	0.0821	1.34	0.0012	2.25

Table 3: Relative error level of a multiscale decomposition at different scales of the trace of an incoming wave

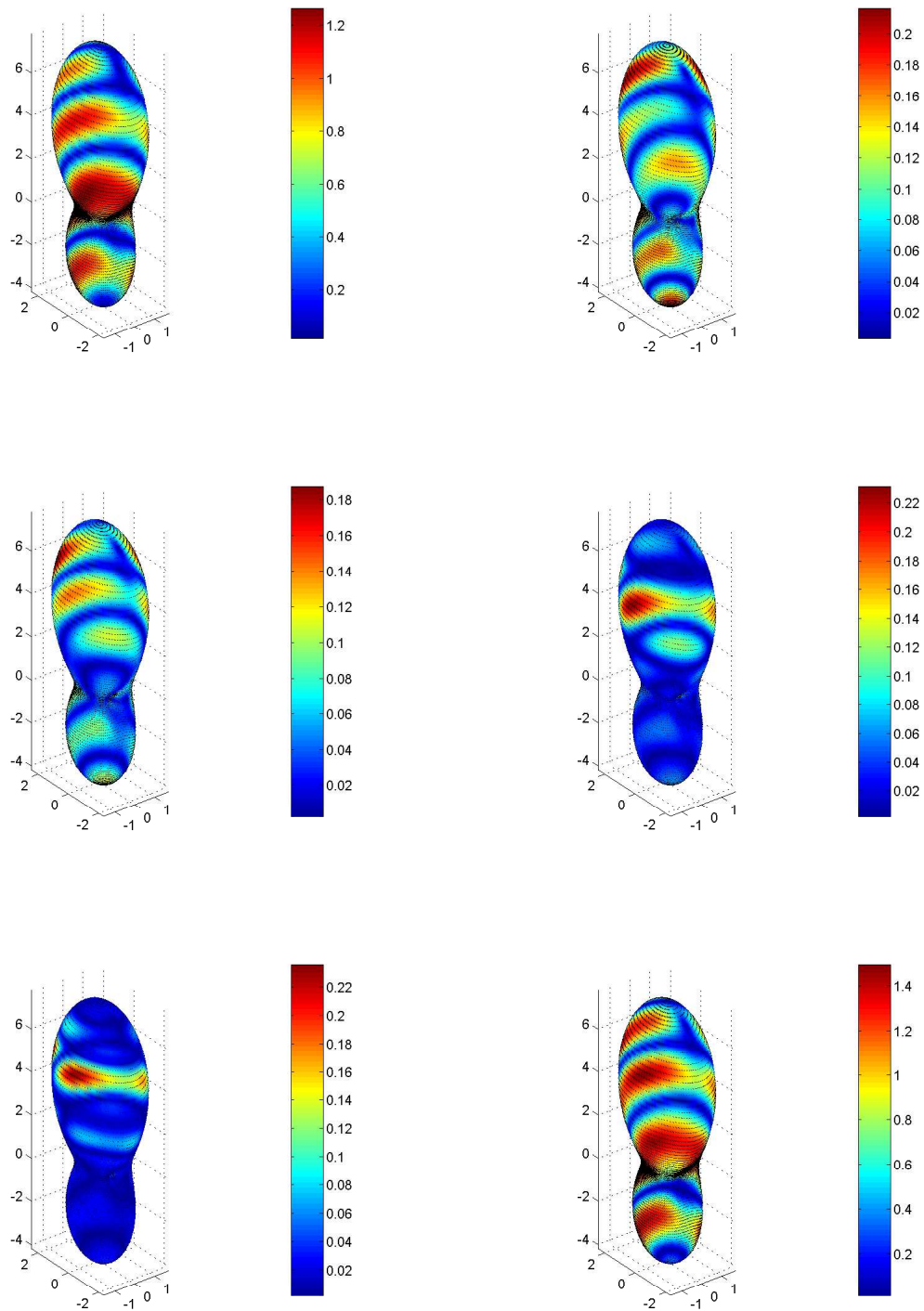


Figure 9: Discrete multiscale decomposition of an incoming wave field on the regular surface defined by (16). The upper left image shows the scale reconstruction at scale $J = 2$. Then the detail reconstructions at the following consecutive scales (from scale $J = 2$ to scale $J = 5$) are presented. Adding them all up results in the scale reconstruction at scale $J = 6$ which can be found in the lower right figure.

6 Multiscale Modelling of Boundary-Value Problems

The aim of this chapter is to introduce a multiscale method for a class of vectorial boundary integral equations over a regular surface Σ . Our equations will be of the form (see, for example, [19])

$$\mu f + Q(0, 0; k)f = g, \quad g \in l_{tan}^2(\Sigma), \quad (17)$$

where $\mu \in \{0, 1, -1\}$ and $Q(0, 0; k)$ is chosen to be a linear combination of the operators $P(0, 0; k)$, $P_{curl}(0, 0; k)$, $Q(0, 0; k)$ defined by (3.1), (3.1), (3.1). More explicitly, we let

$$\begin{aligned} Q(0, 0; k)f(x) = & a \int_{\Sigma} G_k(x, y; k)f(y) d\omega(y) \\ & + b \left(\nu(x) \wedge \int_{\Sigma} f(y) \wedge (\nabla_y G_k)(x, y; k) d\omega(y) \right) + c \left(\nu(x) \wedge \int_{\Sigma} (\nabla_y \otimes \nabla_y + I) G_k(x, y; k) d\omega(y) \right), \end{aligned}$$

with $a, b, c \in \mathbb{C}$. In multiscale nomenclature, the operators $P(0, 0; k)$, $P_{curl}(0, 0; k)$, $Q(0, 0; k)$ can be approximated in the form

$$\begin{aligned} P(0, 0; k)f(x) & \approx \int_{\Sigma} \Phi_{\tau_L}^2(x, y)f(y) d\omega(y) \\ P_{curl}(0, 0; k)f(x) & \approx \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_L}^6(x, y) \wedge f(y) d\omega(y) \\ Q(0, 0; k)f(x) & \approx \nu(x) \wedge \int_{\Sigma} \Phi_{\tau_L}^8(x, y)f(y) d\omega(y) \end{aligned}$$

for $x \in \Sigma$ and sufficiently large $L \in \mathbb{N}$. This results in a multiscale approximation $Q_L(0, 0; k)$ of the operator $Q(0, 0; k)$ of the form

$$\begin{aligned} Q_L(0, 0; k)f(x) = & a \int_{\Sigma} \Phi_{\tau_L}^2(x, y)f(y) d\omega(y) \\ & + b \left(\nu(x) \wedge \int_{\Sigma} \varphi_{\tau_L}^6(x, y) \wedge f(y) d\omega(y) \right) + c \left(\nu(x) \wedge \int_{\Sigma} \Phi_{\tau_L}^8(x, y)f(y) d\omega(y) \right), \end{aligned}$$

for $x \in \Sigma$. In doing so the boundary integral equation of singular type is replaced by a resulting boundary integral equation of regular type. In other words, *regularization of a singular boundary integral equation* is guaranteed by a suitable convolution (low-pass filtering) of the vector field $f \in l_{tan}^2(\Sigma)$ against a *regular kernels* with L sufficiently large.

In accordance with the reconstruction formula (Theorem 4.8) it follows for the unknown function $f \in l_{tan}^2(\Sigma)$ in (17) that

$$f(x) \approx \nu(x) \wedge \int_{\Sigma} \varphi_{\tau_J}^5(x, y)f(y) d\omega(y)$$

provided that J is sufficiently large. Applying an appropriate numerical integration rule with weights $\hat{w}_l^{N_J} \in \mathbb{C}$ and knots $\hat{y}_l^{N_J} \in \Sigma$, we find by letting $\hat{a}_l^{N_J} = \hat{w}_l^{N_J} f(\hat{y}_l^{N_J})$, $l = 1, \dots, N_J$,

$$f(x) \approx \hat{P}_{\tau_J}^i(f)(x) = \nu(x) \wedge \sum_{l=1}^{N_J} \Phi_{\tau_J}^5(x, \hat{y}_l^{N_J}) \wedge \hat{a}_l^{N_J}, \quad x \in \Sigma.$$

The coefficients $\hat{a}_l^{N_J}$, $l = 1, \dots, N_J$, are deducible from the regularized boundary integral equation

$$\nu(x) \wedge \left[\sum_{l=1}^{N_J} \left(\mu \varphi_{\tau_J}^5(x, \hat{y}_l^{N_J}) + \left(Q_L(0, 0; k) \varphi_{\tau_J}^5(\cdot, \hat{y}_l^{N_J}) \right)(x) \right) \wedge \hat{a}_l^{N_J} \right] \approx g(x), \quad x \in \Sigma,$$

by solving a system of linear equations by use of an approximation method (such as collocation, Galerkin approximation, least squares approximation, etc.). For solving the linear system fast multipole methods

(FMM) are applicable (see, e.g., [15]).

The aforementioned steps concerning the boundary integral equation (17) lead us to a tree algorithm with the following ingredients: Starting from $\hat{a}^{N_J} = (\hat{a}_1^{N_J}, \dots, \hat{a}_{N_J}^{N_J})^T$, the coefficients

$$\hat{a}^{N_j} \in \mathbb{C}^{N_j} \times \mathbb{C}^3, \quad \hat{a}^{N_j} = (\hat{a}_1^{N_j}, \dots, \hat{a}_{N_j}^{N_j})^T, \quad j = J_0, \dots, J-1,$$

are determined by the following rules:

- (i) The vectors $\hat{a}^{N_j}, j = J_0, \dots, J-1$, are given recursively (see Chapter 4)

$$\hat{a}_k^{N_j} = \hat{w}_k^{N_j} \sum_{l=1}^{N_j+1} \Xi_j(\hat{y}_k^{N_j}, \hat{y}_l^{N_{j+1}}) \hat{a}_l^{N_{j+1}}.$$

- (ii) For $j = J_0, \dots, J$

$$\hat{P}_{\tau_j}^5(f)(x) \approx \nu(x) \wedge \sum_{k=1}^{N_j} \varphi_{\tau_j}^5(x, \hat{y}_k^{N_j}) \wedge \hat{a}_k^{N_j}, \quad x \in \Sigma$$

- (iii) For $j = J_0 + 1, \dots, J$

$$\hat{R}_{\tau_j}^5(f)(x) \approx \nu(x) \wedge \sum_{k=1}^{N_j} \psi_{\tau_{j-1}}^5(x, \hat{y}_k^{N_j}) \wedge \hat{a}_k^{N_j}, \quad x \in \Sigma$$

Summarizing our results we are finally led to the following multiscale approximation of $f \in l_{tan}^2(\Sigma)$:

$$f_J(x) = \nu(x) \wedge \sum_{l=1}^{N_{J_0}} \Phi_{\tau_{J_0}}^5(x, \hat{y}_l^{N_{J_0}}) \wedge \hat{a}_l^{N_{J_0}} + \nu(x) \wedge \sum_{j=J_0}^J \left(\sum_{l=1}^{N_{j+1}} \Psi_{\tau_j}^5(x, \hat{y}_l^{N_{j+1}}) \wedge \hat{a}_l^{N_{j+1}} \right), \quad x \in \Sigma.$$

It is worth mentioning that the tree algorithm presented above uses an approximation method by a linear system for the initial step and integration rules with (a priori) given weights and knots for the subsequent pyramid steps.

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